

AN EXTENSION AND SOLUTION OF ARROW-KARLIN TYPE PRODUCTION MODELS BY THE PONTRYAGIN MAXIMUM PRINCIPLE

IGAL ADIRI* and ADI BEN-ISRAEL

*Technion-Israel Institute of Technology and Northwestern University***

1. Introduction

The purpose of this note is to show how the Maximum Principle of Pontryagin can be applied to extend and unify the approach of Arrow and Karlin [¹] to problems in Inventory and Production.

To apply the Maximum Principle we must first transform the Arrow-Karlin problems into ones involving differential constraints. The technique is illustrated (after a brief description of the Maximum Principle in § 2) first by deriving the classical lot-size formula [⁵] in this manner. Thereby one may lucidly interpret the available and otherwise unfamiliar dual constructs in the context of a characteristic but simple example. Then the Arrow-Karlin production model of [¹] is extended to the case of non-linear holding costs, and the optimal policy is obtained via the Maximum Principle.

2. The Maximum Principle (¹)

2.1 THE PROBLEM

L.S. Pontryagin and his coworkers [^{3, 11, 4, 8}] developed the Maximum Principle (abbreviated henceforth as M.P.) and applied it to

* The contribution of the first author is part of his M.Sc. dissertation in Operations Research submitted to the Senate of the Technion-Israel Institute of Technology on January 1965.

** Part of the research underlying this report was undertaken for the Office of Naval Research, Contract Nonr-1228(10), Project NR 047-021, and for the U.S. Army Research Office-Durham, Contract No. DA-31-124-ARO-D-322 at Northwestern University. Reproduction of this paper in whole or in part is permitted for any purpose of the United States Government.

(¹) This section is based on [^{2, 9}].

the solution of optimization problems with differential constraints, arising in Control Theory. A representative problem of that sort is the following:

Given the functions:

$$f_i(\bar{x}, \bar{u}) \quad (i = 1, 2, \dots, n)$$

defined and continuous for all vectors

$\bar{x} = (x_1, x_2, \dots, x_1, \dots, x_n)$ in the n -dimensional real vector space E^n and

$\bar{u} = (u_1, u_2, \dots, u_k, \dots, u_r)$ restricted to a certain subset U of E^r .

Furthermore, assume the partial derivatives

$$\frac{\partial f_i(\bar{x}, \bar{u})}{\partial x_j} \quad \begin{array}{l} (i = 1, 2, \dots, n) \\ (j = 1, 2, \dots, n) \end{array}$$

are defined and continuous in $E^n \times U$.

The problem is to determine a piecewise continuous vector function

$$\bar{u} = \bar{u}(t), \quad 0 \leq t \leq T$$

whose values are constrained to lie in U , so that the functions

$$x_1(t), \dots, x_n(t)$$

defined by

$$\dot{x}_i(t) = \frac{dx_i(t)}{dt} = f_i(\bar{x}(t), \bar{u}(t)) \quad (i = 1, \dots, n) \quad (2.1)$$

and by the initial conditions

$$x_i(0) = x_i^0 \quad (2.2)$$

yield a minimum to the objective function

$$S = \int_0^T f_0(\bar{x}(t), \bar{u}(t)) dt \quad (2.3)$$

where $f_0(\bar{x}, \bar{u})$ is defined and continuous in $E^n \times U$ together with its derivatives

$$\frac{\partial f_0(\bar{x}, \bar{u})}{\partial x_i} \quad (i = 1, 2, \dots, n).$$

Remarks:

a) In what follows, the discussion will be restricted to the objective function

$$S = \sum_{i=1}^n c_i x_i(T) \quad (2.4)$$

where the c_i are given constants.

b) If the functions f_i , ($i = 1, \dots, n$), depend explicitly on t - so that the differential equations (2.1) are non-autonomous, the problem can still be reduced to the above form by defining an extra variable x_{n+1} by:

$$\dot{x}_{n+1}(t) = 1, \quad x_{n+1}(0) = 0.$$

c) Using the common terminology we call:

x_1, \dots, x_n the *state variables*

u_1, \dots, u_r the *control variables*

and \bar{u} satisfying $\bar{u}(t) \in U$, $0 \leq t \leq T$, an *admissible control*.

2.2 THE MAXIMUM PRINCIPLE

The M.P. provides a necessary condition for the optimality of $\bar{u}(t)$.

Let the Hamiltonian function

$$H = H(\bar{x}, \bar{u}, \bar{p}) = \sum_{i=1}^n p_i f_i(\bar{x}, \bar{u}) \quad (2.5)$$

be introduced, where the variables

$$\bar{p} = (p_1(t), \dots, p_n(t))$$

satisfy

$$\dot{p}_i = - \sum_{j=1}^n p_j \frac{\partial f_j}{\partial x_i} \quad (i = 1, \dots, n) \quad (2.6)$$

and

$$p_i(T) = -c_i \quad (i = 1, \dots, n). \quad (2.7)$$

Using (2.5), equations (2.1) and (2.6) can be rewritten in the canonical form

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \quad (i = 1, \dots, n) \quad (2.8)$$

$$\dot{p}_i = - \frac{\partial H}{\partial x_i} \quad (i = 1, \dots, n). \quad (2.9)$$

For fixed vectors \bar{x} and \bar{p} let

$$M(\bar{x}, \bar{p}) = \sup_{\bar{u} \in U} H(\bar{x}, \bar{p}, \bar{u}). \quad (2.10)$$

Pontryagin's M.P. states that for an admissible $\bar{u}^*(t)$ and the resulting $\bar{x}^*(t)$ to be optimal it is necessary that there exist a continuous vector function $\bar{p}(t)$ satisfying (2.6) and (2.7) such that :

$$H(\bar{x}^*(t), \bar{u}^*(t), \bar{p}(t)) = M(\bar{x}^*(t), \bar{u}^*(t)) \quad (2.11)$$

for all t in $[0, T]$.

Remarks:

a) While satisfying the needs of this paper, the above is probably the most elementary form of the M.P.. The M.P. was extended to problems with constraints on the final time T and state $\bar{x}(T)$ [12], stochastic problems [12, 10], and others.

b) The relations between the M.P. and Bellman's Dynamic Programming are given in [12, 2, 9, 6, 7].

c) The relations between the M.P. and calculus of variations are given in [12, 9] (1).

3. A Simple Illustration in Inventory Theory

Differential constraints are explicit in the formulation of problems solvable by the M.P.. In other problems, differential constraints can be introduced artificially. We illustrate this by the simplest problem of Inventory Theory, that of determining the optimal lot size under the following assumptions:

- i) The demand rate, r , is constant;
- ii) Demand is instantaneously satisfied;
- iii) A constant amount, Q (the *lot size*) is ordered at fixed time intervals, $T = Q/r$ (the *period*);
- iv) The costs involved are constant:
 - h the cost of storing one inventory unit for one time unit,
 - A the set up cost.

(1) See also the review of [13] in *Math. Rev.*, **22** (1961), 11193.

The solution of this problem, using freshman calculus, can be found in [5]. To solve it by the M.P., the problem is formulated as follows:

$$\text{minimize } S = \sum_{i=1}^2 c_i x_i(T) \quad (3.1)$$

where

$$\dot{x}_1 = 1, \quad x_1(0) = 0, \quad c_1 = 0 \quad (3.2)$$

$$\dot{x}_2 = h \left(1 - \frac{rx_1}{Q} \right) + \frac{Ar}{Q^2}, \quad x_2(0) = 0, \quad c_2 = r. \quad (3.3)$$

Indeed, from (3.1), (3.2) and (3.3) S is found to be:

$$S = \frac{Ar}{Q} + \frac{hQ}{2},$$

the average total cost.

The Hamiltonian, by (2.5), is:

$$H = H(x_1, x_2, p_1, p_2, Q) = p_1 + p_2 \left[h \left(1 - \frac{rx_1}{Q} \right) + \frac{Ar}{Q^2} \right] \quad (3.4)$$

where, by (2.6) and (2.7):

$$\dot{p}_1 = p_2 \frac{hr}{Q}, \quad p_1(T) = 0 \quad (3.5)$$

$$\dot{p}_2 = 0, \quad p_2(T) = -r \quad (3.6)$$

For a control variable Q^* to be optimal, it is necessary by the M.P. that:

$$\left. \frac{\partial H}{\partial Q} \right|_{Q=Q^*} = 0. \quad (3.7)$$

Using (3.4) and (3.7), Q^* must satisfy:

$$p_2 \left[\frac{hrx_1}{Q^2} - \frac{2Ar}{Q^3} \right] = 0. \quad (3.8)$$

Since $p_2 \neq 0$, by (3.6), the only bounded solution of (3.8) is:

$$Q^* = \frac{2A}{hx_1} \quad (3.9)$$

which, for $x_1 = T$, assumes the well known form [5]:

$$Q^* = \sqrt{\frac{2Ar}{h}}. \quad (3.10)$$

4. The Arrow-Karlin Production Model

4.1 THE PROBLEM

In the model of Arrow and Karlin [1], (abbreviated henceforth as the *A.K. model*), the problem is to determine the optimal rate of production $z(t)$, as a function of the time t in the interval $[0, T]$, so as to minimize the total cost.

Assumptions

a) The costs involved are:

$c(z)$ cost of production per time unit when the rate of production is z .

$h(y)$ cost of storing y inventory units for a time unit.

Thus the total cost functional to be minimized is:

$$J(z) = \int_0^T \{c(z(t)) + h(y(t))\} dt \quad (4.1)$$

where

$y(t)$ is the inventory level at time t .

b) The demand is instantaneously satisfied, i.e.:

$$\int_0^t z(\tau) d\tau + y(0) \geq \int_0^t r(\tau) d\tau \quad (0 \leq t \leq T) \quad (4.2)$$

where

$r(t)$ is the rate of demand at time t , assumed known for $0 \leq t \leq T$.

c) The function $c(z)$ is twice differentiable, increasing and strictly convex.

Thus for every $z \geq 0$:

$$\frac{dc}{dz} > 0 \quad \text{and} \quad \frac{d^2c}{dz^2} > 0 \quad (1).$$

d) The function $h(y)$ is a differentiable increasing function.

(1) i.e. the marginal cost dc/dz is a positive increasing function.

Thus the problem is to find a function $z(t)$, t in $[0, T]$, which minimizes:

$$J(z) = \int_0^T \{c(z(t)) + h(y(t))\} dt \text{ so that} \quad (4.1)$$

$$y(t) = y(0) + \int_0^t \{z(\tau) - r(\tau)\} d\tau \geq 0 \quad (4.2)$$

and

$$z(t) \geq 0 \text{ for all } t \text{ in } [0, T]. \quad (4.3)$$

4.2 THE SOLUTION BY THE M.P.

The state variables will be defined as

$$\dot{x}_1(t) = z(t) - r(t), \quad x_1(0) = y(0) \quad (4.4)$$

$$\dot{x}_2(t) = c(z(t)) + h(x_1(t)), \quad x_2(0) = 0$$

Thus $x_1(t) = y(t)$, and since only policies for which

$$y(T) = 0 \quad (4.5)$$

should be considered, an arbitrary value can be assigned to c_1 .

From (4.1) it follows that $c_2 = 1$. With $z(t)$ as the control variable, the functional to be minimized is

$$S(x_1, x_2, z) = \sum_{i=1}^2 c_i x_i(T) = J(z) \quad (4.6)$$

and by (2.5), (4.4) the Hamiltonian is

$$H(x_1, x_2, p_1, p_2, z) = p_1(t) [z(t) - r(t)] + p_2(t) [c(z(t)) + h(x_1(t))] \quad (4.7)$$

where by (2.6) and (4.4):

$$\dot{p}_1 = -p_2 \frac{dh(x_1)}{dx_1} \quad (4.8)$$

$$\dot{p}_2 = 0 \quad (4.9)$$

Using (2.7) and (4.9):

$$p_2(t) = p_2(T) = -c_2 = -1 \text{ for all } t \text{ in } [0, T]. \quad (4.10)$$

Thus, from (4.8) and (4.10):

$$p_1(t) = \int_{t_1}^t \frac{dh(x_1)}{dx_1} d\tau \quad (4.11)$$

and by the arbitrariness of c_1 , $p_1(T)$ is not constrained by (2.7) but rather indirectly by (4.5).

From the M.P. it follows that in any subinterval $[a, b]$ of $[0, T]$, in which:

$$\frac{\partial H}{\partial z} = p_1 + p_2 \frac{d(c(z))}{dz} \quad (4.12)$$

vanishes, the optimal $z_0(t)$ satisfies

$$p_1 + p_2 \frac{d(c(z))}{dz} \Big|_{z=z_0} = 0. \quad (4.13)$$

Using (4.13), (4.10) and (4.11) one obtains:

$$\frac{d(c(z))}{dz} \Big|_{z=z_0} = \int_{t_1}^{t'} \frac{d(h(y))}{dy} d\tau \quad (4.14)$$

where t_1 is determined by the constraints (4.5), (4.2) and (4.3).

Now, assumption *c*) of section 4.1 implies that the function

$$\frac{d(c(z))}{dz} = g(z)$$

is strictly increasing, and therefore the inverse function g^{-1} can be used to determine z_0 from (4.14) and (4.11).

$$z_0(t) = g^{-1}(p_1(t)) = g^{-1} \left[\int_{t_1}^{t'} \frac{dh(y)}{dy} d\tau \right]. \quad (4.15)$$

By (4.12) and assumption *c*) of section 4.1:

$$\frac{\partial^2 H}{\partial z^2} = - \frac{d^2 c}{dz^2} < 0 \quad (4.16)$$

therefore $z_0(t)$ maximizes H for all t in $[a, b]$, which by the M.P. is a necessary condition for the functional (4.6) to be minimized. In any subinterval $[c, d]$ of $[0, T]$ in which $\partial H/\partial z \neq 0$, the optimal z_0 is determined by satisfying the necessary condition that H be maximized.

$\partial H/\partial z \neq 0$ is possible in 2 cases:

- A) $\frac{\partial H}{\partial z} > 0$ in $[c, d]$
- B) $\frac{\partial H}{\partial z} < 0$ in $[c, d]$.

In both cases (4.16) holds.

Case A: From (4.12) and (4.10)

$$\frac{\partial H}{\partial z} = p_1 - \frac{d(c(z))}{dz} > 0. \quad (4.17)$$

Therefore

$$z(t) < g^{-1}(p_1(t)) \text{ for all } t \text{ in } [c, d]. \quad (4.18)$$

Since H is an increasing function, the optimal z_0 has to be taken as large as possible, but for $z_0(t) = g^{-1}(p_1(t))$, $\partial H/\partial z$ vanishes — by (4.12) — contradicting $\partial H/\partial z > 0$ in $[c, d]$. Therefore case A is impossible for the optimal $z_0(t)$.

Case B: With the inequalities (4.17), (4.18) reversed the optimal z_0 has to be taken as small as possible. It is convenient to introduce here the cumulative functions:

$$Z(t) = \int_0^t z(\tau) d\tau$$

$$s(t) = \int_0^t r(\tau) d\tau - y(0)$$

and by the constraints (4.2), (4.3) the optimal z_0 in $[c, d]$ is:

$$Z_0(t) = \max \{Z(c), s(t)\}. \quad (4.19)$$

Remark:

a) It can be shown that in the case of linear $h(y)$:

$$h(y) = hy, \quad h > 0$$

the above results agree with those of Arrow and Karlin [¹] in determining the same optimal policy $z_0(t)$.

b) From the impossibility of case A above, it follows that:

$$\left. \frac{\partial H}{\partial z} \right|_{z=z_0} \leq 0 \text{ in } [0, T].$$

c) In the important special case where $c(z)$ is linear, dc/dz is a positive constant and, by (4.12),

$$\left. \frac{\partial H}{\partial z} \right|_{z=z_0} < 0$$

in $[0, T]$. The optimal policy is then

$$Z_0(t) = \max \{0, s(t)\} \quad (4.20)$$

i.e.

$$z_0(t) = \begin{cases} 0 & \text{if } \int_0^t r(\tau) d\tau \leq y(0) \\ r(t) & \text{if } \int_0^t r(\tau) d\tau > y(0) \end{cases} \quad (4.21)$$

4.3 THE A.K. MODEL WITH SET-UP COSTS AND $r(t) > 0$

The A.K. model, described by assumptions:

a) - d) of section 4.1, will be considered now with 2 additional assumptions:

e) For every start (by *start* we mean a change of the production rate from zero to a positive level) a fixed expense A (*set up cost*) is incurred.

f) The demand rate $r(t)$ is positive during the whole period $[0, T]$.

The model described by assumptions a) - d) of section 4.1 and assumptions e), f) above, will be denoted by the *A.K. set-up model*. It will now be shown that the optimal policy is the *same* in both the A.K. model and the A.K. set-up model:

In the A.K. set-up model the cost functional is:

$$J(z, n) = nA + \int_0^T \{c(z(t)) + h(y(t))\} dt \quad (4.22)$$

where

n is the number of starts, and $y(t)$, $z(t)$ are constrained by (4.2), (4.3) as before.

The following notations are now introduced:

z_{01} the optimal policy in the A.K. set-up model (recall that z_0 denotes the optimal policy in the A.K. model).

n_0 the number of starts in the policy z_0 .

n_{01} the number of starts in the policy z_{01} .

The policy z_{01} is different from the policy z_0 only if the number of starts n_0 can be decreased, i.e. only if $n_0 > 1$ ⁽¹⁾. In order that $n_0 > 1$

⁽¹⁾ Clearly, if $\int_0^T r(\tau) d\tau > y(0)$, which is the only case of interest, then $n_0 \geq 1$.

it is necessary that $z_0 = 0$ in a subinterval of $[0, T]$, following a subinterval in which $z_0 > 0$. We shall prove that this is impossible:

Suppose a subinterval of $[0, T]$ in which $z_0 = 0$ follows a subinterval in which $z_0 > 0$, i.e. either

$$z_0(t) = g^{-1} \left[\int_{t_1}^t \frac{dh(y)}{dy} d\tau \right] \quad (4.15)$$

or [from (4.19)]

$$z_0(t) = r(t). \quad (4.23)$$

Both cases are impossible:

Case (4.15):

A subinterval in which

$$z_0(t) = g^{-1} \left[\int_{t_1}^t \frac{dh(y)}{dy} d\tau \right] > 0 \quad (4.24)$$

is followed by a subinterval in which $z_0(t) = 0$, i.e. in which $\partial H/\partial z < 0$ (see case B, section 4.2) i.e. in which [from (4.12), (4.10) and (4.11)]:

$$g^{-1} \left[\int_{t_1}^t \frac{dh(y)}{dy} d\tau \right] < 0. \quad (4.25)$$

But a subinterval with (4.25) cannot follow a subinterval with (4.24), because the functions g, h are monotone increasing.

Case (4.23):

Let $z_0(t) = r(t) > 0$ in $[c, d]$ and $z_0(t) = 0$ in $[d, e]$ where $0 \leq c \leq e \leq T$. In $[c, e]$ $\partial H/\partial z < 0$ and z_0 is determined by

$$Z_0(t) = \max \{Z(c), s(t)\}. \quad (4.19)$$

But this means that:

$$\begin{aligned} Z(c) &< s(t) \text{ in } [c, d] \\ Z(c) &> s(t) \text{ in } [d, e] \end{aligned}$$

which contradicts $r(t) > 0$.

Therefore, the optimal policy is the same for both the A.K. model and the A.K. set-up model:

$$z_0(t) = z_{01}(t)$$

and

$$\min J(z, n) = J(z_0, 1) = J(z_0) + A.$$

