

An inverse Newton transform

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Dedicated to the memory of Dan Butnariu

ABSTRACT. Given a function $u : \mathbb{R} \rightarrow \mathbb{R}$, the inverse Newton transform of u , denoted $\mathbf{N}^{-1}u$, is the function

$$f(x) = \exp \left\{ \int \frac{dx}{x - u(x)} \right\},$$

wherever existing. The iterations $x := u(x)$ coincide with the Newton iterates for $\mathbf{N}^{-1}u$, and in this sense every iteration (for which $\mathbf{N}^{-1}u$ exists) is Newton. The correspondence $u \longleftrightarrow f = \mathbf{N}^{-1}u$ may be useful since the zeros of f are fixed points of u , and with continuous differentiability of u , zeros of order $> \frac{1}{2}$ are attracting fixed points. Consequently, standard existence and convergence results for Newton's method, and its geometric interpretation, may apply to general iterations in the domain of their inverse Newton transform.

1. Introduction

The **Newton transform** $\mathbf{N}f$ of a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ is the function,

$$(1.1) \quad (\mathbf{N}f)(x) := x - \frac{f(x)}{f'(x)},$$

provided $f'(x) \neq 0$. The **inverse Newton transform** of a function $u : \mathbb{R} \rightarrow \mathbb{R}$, denoted $\mathbf{N}^{-1}u$, is a function f (necessarily differentiable) such that $\mathbf{N}f = u$, or,

$$(1.2) \quad x - \frac{f(x)}{f'(x)} = u(x),$$

for x in an appropriate domain. Relations between u and $\mathbf{N}^{-1}u$ were studied by Gorn [4].

The inverse Newton transform is defined up to a nonzero constant multiple, since $\mathbf{N}f = \mathbf{N}(cf)$ for all $c \neq 0$. It is not defined for the identity $u(x) = x$.

The inverse Newton transform is often computable in closed form, as illustrated here with the symbolic algebra package `Maple`, [6]. The correspondence between

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u and $\mathbf{N}^{-1}u$ allows applying standard results for Newton's method to general iterations. In particular, this correspondence may explain the behavior of iterations around their fixed points, as illustrated here for the logistic iteration, see Section 4.

The main results, collected in Section 2 are:

Theorem 2.1: Relations between the zeros of $N^{-1}u$ and the fixed points of u .

Theorem 2.4: An integral form and properties of the inverse Newton transform.

Corollary 2.7: Existence of attracting fixed points, and quadratic convergence.

The results are illustrated by examples in Section 3, and the logistic iteration in Section 4. Complex iterations are considered in Section 5.

2. Results

We call ζ a **zero** of f of **order** m if

$$(2.1) \quad f(x) = (x - \zeta)^m g(x)$$

where m is a positive real, and $g(\zeta) \neq 0$.

Next, some consequences of the above definitions, for $u = \mathbf{N}f$:

(a) If f is twice differentiable, then by (1.2),

$$(2.2) \quad u'(x) = \frac{f(x)f''(x)}{f'(x)^2}.$$

(b) If ζ is a zero of f of order m , then

$$(2.3) \quad u'(x) = \frac{f(x)f''(x)}{f'(x)^2} = \frac{m(m-1)g(x)^2 + 2(x-\zeta)g'(x) + (x-\zeta)^2g''(x)}{m^2g(x)^2 + 2(x-\zeta)mg'(x) + (x-\zeta)^2g'(x)^2} \rightarrow \frac{m-1}{m},$$

as $x \rightarrow \zeta$, provided $\lim_{x \rightarrow \zeta} (x - \zeta)g'(x) = \lim_{x \rightarrow \zeta} (x - \zeta)^2g''(x) = 0$.

(c) If ζ is a zero of f of order $m < 1$, then f is not differentiable at ζ , but u may still be defined and differentiable at ζ , in which case $u'(\zeta) = \frac{m-1}{m}$, see Example 2.3.

Next, some relations between the zeros of f and the fixed points of u , see also [1, §1.9].

THEOREM 2.1. *Let $f = \mathbf{N}^{-1}u$, and let ζ be a point where f is differentiable. In parts (a), (b), and (c) assume $f'(\zeta) \neq 0$.*

(a) ζ is a zero of f if, and only if, it is a fixed point of u .

(b) If ζ is a zero of f , f and u are twice differentiable at ζ , then ζ is a superattracting fixed point of u , and convergence is (at least) quadratic.

(c) If ζ is a zero of f of order $m > \frac{1}{2}$, and u is continuously differentiable at ζ , then ζ is an attracting fixed point of u .

(d) Let ζ have a neighborhood where u and f are continuously differentiable, and $f'(x) \neq 0$ except possibly at $x = \zeta$. If ζ is an attracting fixed point of u then it is a zero of f .

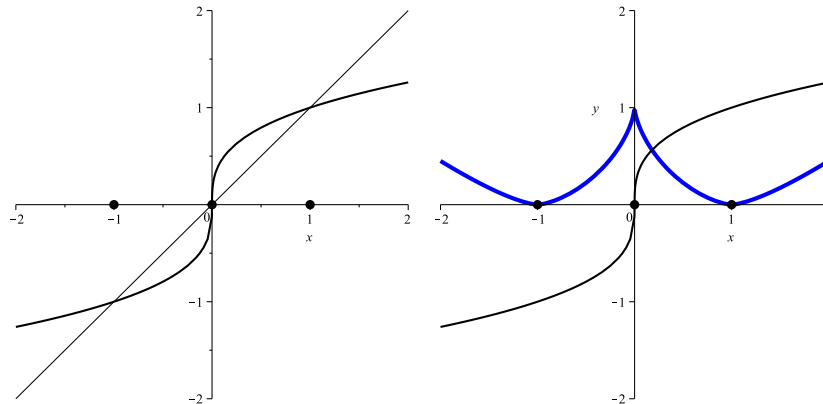
PROOF.

(a) Follows from (1.2).

(b) If $f(\zeta) = 0$ then $u'(\zeta) = 0$ by (2.2), and

$$\frac{u(x) - \zeta}{(x - \zeta)^2} \rightarrow \frac{u''(\zeta)}{2!} \quad \text{as } x \rightarrow \zeta.$$

(c) If u' is continuous at $x = \zeta$, then by (2.3), $u'(\zeta) = \frac{m-1}{m}$. Therefore $|u'(\zeta)| < 1$ if $m > \frac{1}{2}$.



(a) The 3 fixed points of $x^{1/3}$ at $0, \pm 1$ (b) The inverse Newton transform (2.5)

FIGURE 1. The fixed points and inverse Newton transform of the iteration $x := x^{1/3}$

(d) Since ζ is an attracting fixed point of u , $\lim_{x \rightarrow \zeta} u(x) = \zeta$, and by (1.2), $\lim_{x \rightarrow \zeta} \frac{f(x)}{f'(x)} = 0$. In other words, for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x - \zeta| < \delta \implies |f(x)| < \epsilon |f'(x)|,$$

and $f(\zeta) = 0$, by the continuity of f' . \square

The Newton transform can be computed symbolically, for example using **Maple**,

(2.4) `Newton:=proc(f,x); x-f/diff(f,x); end:`

entered after the **Maple** prompt `>`, [6].

EXAMPLE 2.2. The inverse Newton transform of $u(x) = x^{1/3}$ is,

(2.5)
$$\mathbf{N}^{-1}(x^{1/3})(x) = (x^{2/3} - 1)^{3/2}$$

as can be verified directly, or by using (2.4),

`> simplify(Newton((x^(2/3)-1)^(3/2),x));`

giving

$$x^{1/3}$$

Fig. 1(a) shows the 3 fixed points of $x := x^{1/3}$, two attracting fixed points at ± 1 , and one repelling fixed point at 0. The inverse Newton transform (2.5) is shown in Fig. 1(b). It is convex in $(-\infty, 0)$ and in $(0, \infty)$, by Theorem 2.4(c) below. The attracting fixed points ± 1 are zeros of (2.5), in accordance with Theorem 2.1(d). The repelling fixed point 0 is not a zero of (2.5), but a pole of its derivative.

The derivative u' exists at $x = \pm 1$, but is not given there by (2.2) because f is not twice differentiable at $x = \pm 1$. In particular, $u'(\pm 1) \neq 0$ although $f(\pm 1) = 0$.

To plot Fig. 1(b) it may be necessary to write RHS(2.5) as

$$\text{surd}(\text{surd}(x,3)^2-1,2)^3$$

(the **Maple** command `surd(x,n)` computes the n _{th} root of the complex number x whose (complex) argument is closest to that of x .)

EXAMPLE 2.3. Let $m > 0$ and n be reals, and

$$(2.6) \quad f(x) := x^m (x - 1)^n$$

with a zero of order m at $x = 0$. The Newton transform of f is

$$(2.7) \quad u(x) = \frac{(m + n - 1)x^2 - (m - 1)x}{(m + n)x - m}$$

and is defined and differentiable at $x = 0$ (even for $m < 1$ when $f'(0)$ is undefined), with

$$u'(0) = \frac{m - 1}{m}$$

in accordance with (2.3). Therefore $x = 0$ is a fixed point of u that is repelling if $m < 1/2$, indifferent if $m = 1/2$, and attracting if $m > 1/2$.

Next, an integral form for the inverse Newton transform.

THEOREM 2.4. Let u be a function: $\mathbb{R} \rightarrow \mathbb{R}$, D a region where

$$\frac{1}{x - u(x)}$$

is integrable. Then in D ,

$$(2.8) \quad (\mathbf{N}^{-1}u)(x) = C \exp \left\{ \int \frac{dx}{x - u(x)} \right\}, \quad C \neq 0.$$

Moreover, if $C > 0$ then $\mathbf{N}^{-1}u$ is

- (a) increasing if $x > u(x)$,
- (b) decreasing if $x < u(x)$,
- (c) convex if u is differentiable and increasing, or
- (d) concave if u is differentiable and decreasing.

Note: (2.8) appeared in [4, p. 464], and as an exercise in [10, p. 40].

PROOF. Assuming

$$(2.9) \quad x \neq u(x),$$

rearrange (1.2)

$$(2.10) \quad \frac{f'(x)}{f(x)} = \frac{1}{x - u(x)},$$

and integrate to get

$$\ln f(x) = \int \frac{dx}{x - u(x)} + C,$$

or

$$f(x) = C \exp \left\{ \int \frac{dx}{x - u(x)} \right\},$$

where, without loss of generality, $C = 1$. Differentiating f gives

$$(2.11a) \quad f'(x) = \frac{1}{x - u(x)} \exp \left\{ \int \frac{dx}{x - u(x)} \right\},$$

$$(2.11b) \quad f''(x) = \frac{u'(x)}{(x - u(x))^2} \exp \left\{ \int \frac{dx}{x - u(x)} \right\},$$

which prove statements (a)–(d). \square

Next, the inverse Newton transforms of affine transformations of iterations.

COROLLARY 2.5. *Let $a \neq 0$ and b be reals, and let*

$$(2.12) \quad f := \mathbf{N}^{-1}(u(ax + b)),$$

then

$$(2.13) \quad (\mathbf{N}^{-1}(au + b))(x) = f\left(\frac{x - b}{a}\right)$$

PROOF. To compute $\mathbf{N}^{-1}(au(x) + b)$ write the integral in (2.8) as

$$\begin{aligned} \int \frac{dx}{x - (au(x) + b)} &= \int \frac{dx}{a \left(\left(\frac{x - b}{a} \right) - u(x) \right)} \\ &= \int \frac{dy}{y - u(ay + b)}, \text{ for } x := ay + b. \end{aligned}$$

$$\therefore (\mathbf{N}^{-1}(au(x) + b))(x) = (\mathbf{N}^{-1}u(ax + b))(y),$$

and the result follows by changing from y back to x . \square

Note: If the above affine transformation is denoted by ϕ

$$(2.14) \quad \phi(x) := ax + b,$$

then Corollary 2.5 states that

$$(2.15) \quad \mathbf{N}^{-1}(\phi u) = \phi^{-1} \mathbf{N}^{-1}(u \phi),$$

not to be confused with a statement about conjugates of u .

If u is monotone then the iterations $x := u(x)$ are reversed by $x := u^{-1}(x)$. The next corollary gives the inverse Newton transform of u^{-1} .

COROLLARY 2.6. *Let u be monotone and differentiable, and let,*

$$(2.16) \quad f(x) := \exp \left\{ \int \frac{u'(x) dx}{u(x) - x} \right\}$$

Then the inverse Newton transform of u^{-1} is

$$(2.17) \quad (\mathbf{N}^{-1}(u^{-1}))(x) = f(u^{-1}(x)).$$

Note: The inverse u^{-1} is not involved in the integration (2.16), a possible advantage over the direct computation in (2.18) below.

PROOF. The reverse of the iteration $x_+ := u(x)$ is $x := u^{-1}(x_+)$. The inverse Newton transform of u^{-1} is by (1.2),

$$(2.18) \quad (\mathbf{N}^{-1}(u^{-1}))(x_+) = \exp \left\{ \int \frac{dx_+}{x_+ - u^{-1}(x_+)} \right\}$$

changing variables to $x = u^{-1}(x_+)$ we get

$$(\mathbf{N}^{-1}(u^{-1}))(u(x)) = \exp \left\{ \int \frac{u'(x) dx}{u(x) - x} \right\}$$

proving the corollary. \square

For $f = \mathbf{N}^{-1}u$, the iterations $x := u(x)$ coincide with the Newton iterations on f . It is therefore natural to ask what, if anything, can be learned from Newton's method that is useful for general iterations. One answer is given in the following result, rooted in standard existence and convergence conditions for Newton's method.

COROLLARY 2.7. (After Ostrowski, [8, Theorem 7.2]).

Let u be an iteration function, $f = \mathbf{N}^{-1}u$ its inverse Newton transform, x_0 a point such that

$$f(x_0)f'(x_0) \neq 0,$$

and

$$h_0 := -\frac{f(x_0)}{f'(x_0)}.$$

Consider the interval $J_0 := [x_0, x_0 + 2h_0]$, and assume that $f''(x)$ exists in J_0 , and

$$(2.19a) \quad \max_{x \in J_0} |f''(x)| = M,$$

$$(2.19b) \quad |f'(x_0)| \geq 2|h_0|M.$$

Then the sequence

$$(2.20) \quad x_{k+1} := x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \dots,$$

lies in J_0 , and converges to a point ζ that is the unique fixed point of u in J_0 .

Moreover, if $\zeta \neq x_0 + 2h_0$ then it is a simple zero of f , and convergence is quadratic,

$$(2.21) \quad |x_{n+1} - \zeta| \leq \frac{1}{2} \frac{M}{|f'(x_n)|} |x_n - \zeta|^2.$$

PROOF. A step in the proof, see [8, Theorem 7.2], is that the sequence (2.20) satisfies

$$(2.22) \quad |f'(x_k)| \geq 2|h_k|M, \quad \text{where } h_k = f(x_k)/f'(x_k), \quad k = 0, 1, \dots$$

That ζ is attracting follows from

$$|u'(x_k)| = \left| \frac{f(x_k)f''(x_k)}{f'(x_k)^2} \right| \leq \left| \frac{h(x_k)M}{f'(x_k)} \right| \leq \frac{1}{2}, \quad \text{for all } k,$$

by (2.2) and (2.22), proving that $|u'(\zeta)| < 1$. □

The hypotheses of the corollary are roughly that $|f(x_0)|$ and $|f''(x_0)|$ are not too large, $|f'(x_0)|$ not too small, and $|f''(x)|$ is bounded in the interval of interest. These conditions suffice for the existence of an attracting fixed point, but are not necessary, as shown next.

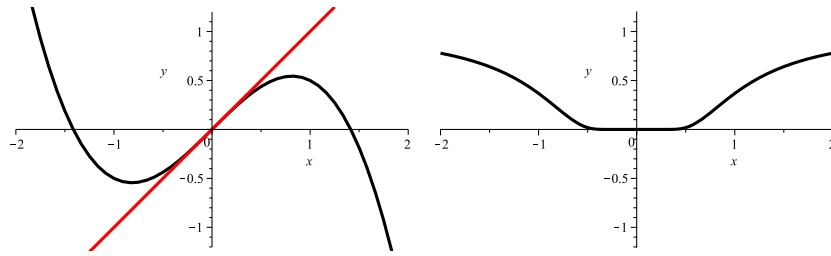
EXAMPLE 2.8. The iteration

$$(2.23a) \quad u(x) := x - \frac{1}{2}x^3$$

has the inverse Newton transform

$$(2.23b) \quad f(x) = \exp\{-1/x^2\},$$

a pathological function that vanishes at $x = 0$, together with all its derivatives.



(a) u has an attracting fixed point at 0 (b) f and all its derivatives vanish at 0

FIGURE 2. The iteration $u(x) = x - \frac{1}{2}x^3$ and its inverse Newton transform $f(x) = \exp\{-1/x^2\}$

The iteration u has an attractive fixed point at $x = 0$, although $u'(0) = 1$, see also [7, Ex. 10.1–2], but convergence is slow (the step at x is $-\frac{1}{2}x^3$.)

Note that $z = 0$ is not an attracting fixed point of the complex iteration

$$u(z) = z - \frac{1}{2}z^3,$$

that takes points along the imaginary axis further away from $z = 0$, see also [1, Lemma 6.5.2].

3. Examples

Symbolic computation. The inverse Newton transform (2.8) is computed in Maple as follows

```
(3.1) InverseNewton:=proc(u,x);simplify(exp(int(1/(x-u),x)));end:
```

Similarly, the inverse transform (2.16) deserves a special name, and is computed by,

```
(3.2) ReverseNewton:=proc(u,x);
      simplify(exp(int(diff(u,x)/(u-x),x)));end:
```

EXAMPLE 3.1. **Affine iterations.** Consider the iteration

$$(3.3) \quad u(x) = ax + b$$

where a, b are constants, not both zero. $\mathbf{N}^{-1}u$ is undefined if $u(x) = x$, i.e., if $a = 1$ and $b = 0$.

If $a = 1$ and $b \neq 0$, (3.3) has no fixed point, and

$$\mathbf{N}^{-1}(x + b) = \exp\left\{-\frac{x}{b}\right\}.$$

If $a \neq 1$, then (3.3) has a single fixed point $x = \frac{b}{1-a}$, and the inverse Newton transform is

$$(3.4) \quad \left(x - \frac{b}{1-a}\right)^{\frac{1}{1-a}}.$$

In particular, the iterations $x := ax$ correspond to Newton iterations on $f(x) = x^{1/1-a}$.

EXAMPLE 3.2. Powers. Let p be a real number, $p \neq 1$. Then the inverse Newton transform of x^p is

$$(3.5) \quad \mathbf{N}^{-1}(x^p)(x) = \frac{x}{(x^{p-1} - 1)^{1/p-1}}$$

In particular,

$u(x)$	$(\mathbf{N}^{-1}u)(x)$
x^2	$x/(x-1)$
1	$x-1$
$x^{1/2}$	$-(\sqrt{x}-1)^2$

EXAMPLE 3.3. Reverse iterations. To illustrate Corollary 2.6 consider $u(x) = x^3$. The inverse Newton transform of the reverse iteration $u^{-1}(x) = x^{1/3}$ is computed by (3.2),

> subs(x=x^(1/3),ReverseNewton(x^3,x));

simplifying to

$$(x^{1/3} - 1)^{3/2} (x^{1/3} + 1)^{3/2}$$

which agrees with (2.5).

EXAMPLE 3.4. Quadratic iterations. Consider the iteration function

$$(3.6) \quad u(x) = ax^2 + bx + c$$

with real coefficients a, b, c . Then

$$\begin{aligned} x - u(x) &= -ax^2 - (b-1)x - c \\ &= -a(x-x_-)(x-x_+), \end{aligned}$$

where x_{\pm} are the roots of $u(x) - x$,

$$(3.7) \quad x_{\pm} = \frac{-(b-1) \pm \sqrt{(b-1)^2 - 4ac}}{2a},$$

that are fixed points of u .

To apply (2.8), compute

$$\begin{aligned} \frac{1}{x-u(x)} &= -\frac{1}{a(x-x_-)(x-x_+)} \\ &= -\frac{1}{a} \left(\frac{A}{x-x_-} - \frac{A}{x-x_+} \right), \quad A = -\frac{1}{x_+ - x_-}. \\ \therefore \int \frac{dx}{x-u(x)} &= \frac{1}{a(x_+ - x_-)} \int \left(\frac{1}{x-x_-} - \frac{1}{x-x_+} \right) dx \\ &= \ln \left(\frac{x-x_-}{x-x_+} \right)^{1/a(x_+ - x_-)} \end{aligned}$$

and, by Theorem 2.4, the inverse Newton transform of (3.6) is

$$(3.8) \quad (\mathbf{N}^{-1}u)(x) = \left(\frac{x - x_-}{x - x_+} \right)^{1/a(x_+ - x_-)}$$

If $(b - 1)^2 - 4ac = 0$ then $x_+ = x_-$, a double root, and (2.8) gives,

$$(\mathbf{N}^{-1}u)(x) = \exp \left\{ \frac{1}{a(x - x_+)} \right\}$$

the limit of (3.8) as $x_+ \rightarrow x_-$.

Assume $(b - 1)^2 - 4ac \geq 0$, in which case the fixed points x_{\pm} are real. Then

$$u'(x_{\pm}) = 1 \pm \sqrt{(b - 1)^2 - 4ac},$$

in particular $u'(x_+) > 1$, i.e., x_+ is a repelling fixed point. The nature of the other fixed point x_- depends on the magnitude of $\sqrt{(b - 1)^2 - 4ac}$.

In the special case

$$(3.9) \quad u(x) = x^2 + c, \quad c \text{ real},$$

we get from (3.8),

$$(3.10) \quad (\mathbf{N}^{-1}(x^2 + c))(x) = \left(\frac{x - x_-}{x - x_+} \right)^{1/(x_+ - x_-)}$$

where $x_{\pm} = \frac{1}{2}(1 \pm \sqrt{1 - 4c})$ are the roots of $x - x^2 - c = 0$, i.e., the fixed points of $x^2 + c$ (that are real if $c \leq \frac{1}{4}$). Therefore, for $c \neq \frac{1}{4}$,

$$(3.11) \quad (\mathbf{N}^{-1}(x^2 + c))(x) = \left(\frac{x - \frac{1}{2}(1 - \sqrt{1 - 4c})}{x - \frac{1}{2}(1 + \sqrt{1 - 4c})} \right)^{1/\sqrt{1 - 4c}}$$

and by taking a limit as $c \rightarrow \frac{1}{4}$,

$$(\mathbf{N}^{-1}(x^2 + \frac{1}{4}))(x) = \exp \left(\frac{2}{2x - 1} \right).$$

EXAMPLE 3.5. Relatives of Newton's method. Consider the class of iterative methods

$$(3.12) \quad u(x) = x - \frac{f(x)}{f'(x) - a(x)f(x)}$$

for an appropriate function $a(x)$. This class includes the Newton method, the Halley method

$$(3.13) \quad u(x) := x - \frac{f(x)}{f'(x) - \frac{f''(x)}{2f'(x)}f(x)},$$

and other useful methods, see, e.g., [2].

The inverse Newton transform of (3.12) is computed by Maple using (3.1),

```
> InverseNewton(x-f(x)/(diff(f(x),x)-a(x)*f(x)),x);
```

giving

$$(3.14) \quad f(x) \exp \left\{ - \int a(x) dx \right\}$$

showing that (3.12) is equivalent to the Newton iteration on (3.14).

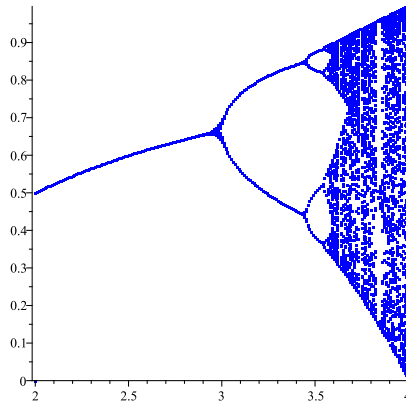


FIGURE 3. 100 iterates of the logistic function (4.2) for selected values of $2 \leq \mu \leq 4$

For the Halley method, with $a(x) = f''(x)/2f'(x)$, (3.14) gives the inverse Newton transform,

$$(3.15) \quad \frac{f(x)}{\sqrt{f'x}}$$

confirming the well known result that the Halley iteration is equivalent to the Newton iteration on the function (3.15).

4. The logistic iteration

The inverse Newton transform helps visualize an iteration

$$(4.1) \quad x := u(x)$$

around its non-attracting fixed points ζ . Indeed, if u is decreasing at an interval containing ζ , then the inverse Newton transform is concave at that interval (by Theorem 2.4(d)), and the iterates may bounce from one side of ζ to the other.

We illustrate this for the **logistic iteration function**

$$(4.2) \quad u(x) = \mu x(1 - x),$$

where μ is a constant parameter, restricted to $0 \leq \mu \leq 4$ (in order to keep the iterates in $[0, 1]$.) The fixed points of (4.2) are 0 and

$$(4.3) \quad \zeta(\mu) = \frac{\mu - 1}{\mu}.$$

The fixed point (4.3) is attracting for $1 < \mu < 3$, and repelling for $\mu > 3$, with bifurcations followed by chaos, see, e.g., [3, Chapter 3]. Fig. 3 is a standard illustration, showing 100 iterates of (4.2), starting at $x_0 = 0.5$, for 200 values of μ (2 to 4 with steps of 0.01), after discarding the first 50 iterates (to make the results independent of x_0 .)

Using the results of Example 3.4 with $a = -\mu$, $b = \mu$, $c = 0$ we conclude that the logistic iteration is equivalent to a Newton iteration with the function

$$(4.4) \quad f(x) = \left(\frac{x - \zeta(\mu)}{x} \right)^{\frac{1}{\mu-1}},$$

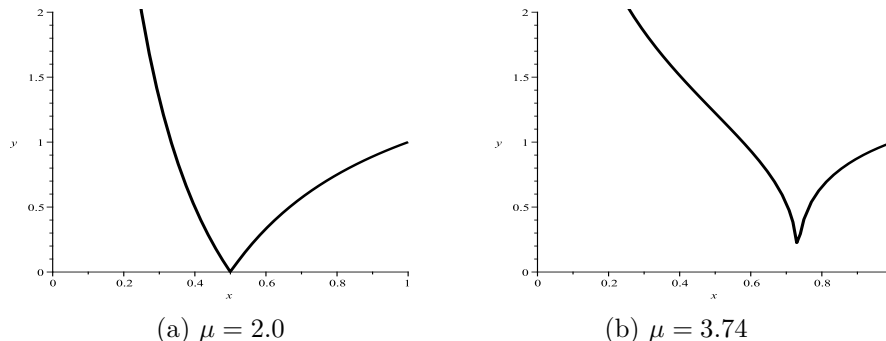


FIGURE 4. The inverse Newton transform of $u(x) = \mu x(1 - x)$

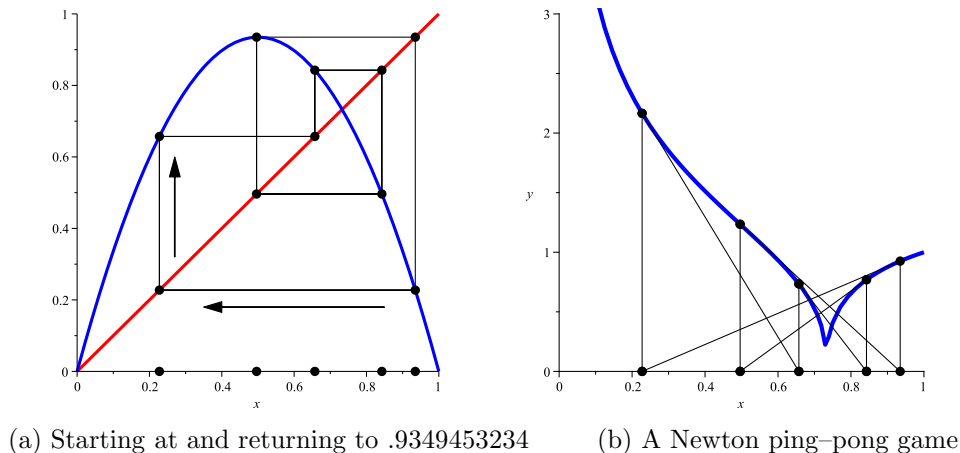


FIGURE 5. The logistic iteration has a 5-cycle for $\mu = 3.74$

which, by Theorem 2.4(c,d) is convex for $0 < x < \frac{1}{2}$, and concave for $\frac{1}{2} < x$. This function is displayed in Fig. 4, for 2 values of μ .

Fig. 4(a) shows the case $\mu = 2.0$, where the inverse Newton transform (4.4) is convex to the left of its zero, and concave to its right. This guarantees global convergence of the Newton iterates of (4.4) to the fixed point $\zeta(2) = 0.5$.

For $\mu = 3.74$, the case shown in Fig. 4(b), the inverse Newton transform (4.4) is concave on both sides of its zero, and the Newton iterates bounce off the point $\zeta(3.74)$, which is thus a repelling fixed point of (4.2). At $\mu = 3.74$, the logistic iterates have a 5-cycle, as shown in Fig. 5(a). The corresponding Newton iterates in Fig. 5(b) offer a new visualization as a ping-pong game between the two sides of a non-attracting fixed point.

5. Complex iterations

The above results carry over, with caution, to analytic functions of complex variables. If u is an analytic function $:\mathbb{C} \rightarrow \mathbb{C}$, then as in the real case, the inverse Newton transform $\mathbf{N}^{-1}u$ is defined by (1.1), and computed by (2.8) or (3.1), where the integral is interpreted as anti-derivative.

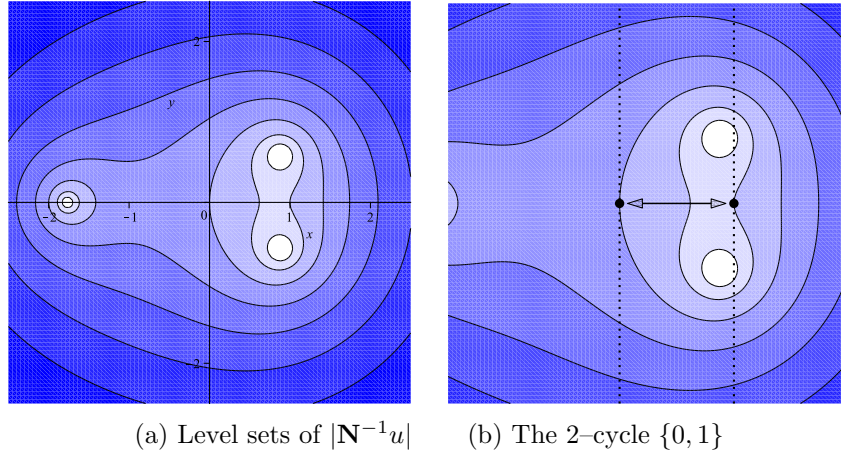
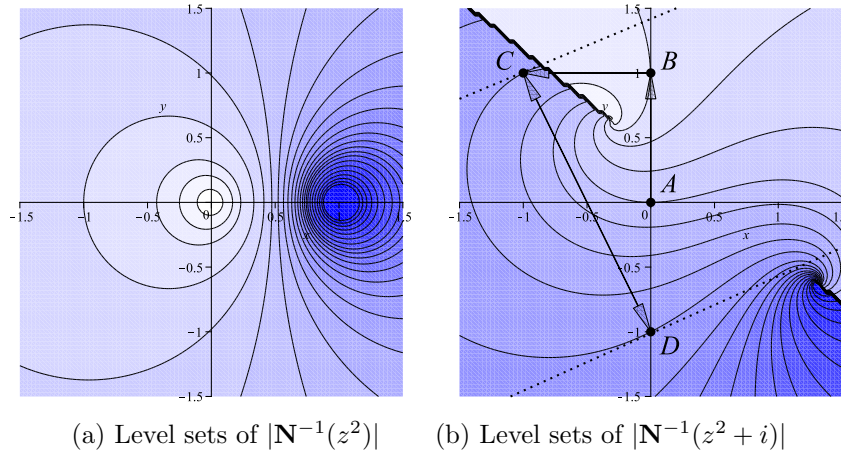


FIGURE 6. Illustration of Example 5.1

FIGURE 7. Level sets of $|\mathbf{N}^{-1}(z^2 + c)|$ for $c = 0$ and $c = i$

As in the real case, we can use the theory of Newton's method to prove existence of attracting fixed points of u , and quadratic convergence, under standard hypotheses. The results are analogous to Corollary 2.7, with a disk in \mathbb{C} replacing the interval J_0 , and the proof of [8, Theorem 7.1] can be used verbatim.

Another advantage of the inverse Newton transform is that it allows a common geometric understanding of all iterations u in terms of the Newton method applied to $\mathbf{N}^{-1}u$.

First, recall the geometric interpretation, [11], of the complex Newton iteration

$$(5.1) \quad z_+ := z - \frac{f(z)}{f'(z)}$$

that takes a point z where $f'(z) \neq 0$ into a point z_+ .

(A) Let $z = x + iy \longleftrightarrow (x, y)$ be the natural correspondence between complex variables $z \in \mathbb{C}$ and points $(x, y) \in \mathbb{R}^2$, and let $F(x, y) := f(z)$ for $z \longleftrightarrow (x, y)$.

(B) Let $T \subset \mathbb{R}^3$ be the plane tangent to the graph of $|F|$ at the point $(x, y, |F(x, y)|)$, and let L be the line of intersection of T and the (x, y) -plane, called the **trace** of T (L is nonempty by the assumption that $f'(z) \neq 0$.)

(C) Then z_+ in (5.1) is $z_+ \longleftrightarrow (x_+, y_+)$, the perpendicular projection of (x, y) on L .

The trace L in (B) above is parallel to the tangent of the level set of $|F(x, y)|$ at the point (x, y) . The complex Newton iterations for f , and the corresponding iterations of $u = \mathbf{N}f$, are thus driven by the level sets of the $|f|$, moving perpendicular to these level sets, against the gradient of $|F|$.

EXAMPLE 5.1. The complex iteration

$$(5.2) \quad u(z) = 2 \frac{z^3 - 1}{3z^2 - 2}$$

has a 2-cycle, $u(0) = 1, u(1) = 0$. The inverse Newton transform of $u(z)$ is

$$(5.3) \quad f(z) = z^3 - 2z + 2$$

as can be verified by Maple,

```
> InverseNewton(2*(z^3-1)/(3*z^2-2), z)
```

Fig. 6(a) shows the level sets of $|f(z)|$, and the approximate locations of the three zeros of $f(z)$,

$$z = -1.769292354, \text{ and } z = .8846459052 \pm .5897426790 i,$$

that are attracting fixed points of $u(z)$. Indeed, near these roots the Newton iterates for (5.3) bounce off the level sets of $|f(z)|$ towards the roots.

Fig. 6(b) is a detail, showing the 2-cycle $\{0, 1\}$, and illustrating the geometric interpretation of (5.1): The points $z = 0$ and $z = 1$ are repeated, since the traces of the tangent planes of the level sets at these points are parallel (see dotted lines in Figure 6(b)), and the step size is just right.

Newton iterates of (5.3) and related cubic polynomials were studied in [9].

EXAMPLE 5.2. We conclude with the complex iteration

$$(5.4) \quad z := z^2 + c, \quad c \text{ constant.}$$

This function figures in the definition of the **Mandelbrot set**, [5],

$$(5.5) \quad \mathcal{M} := \{c : \text{for which the iterates (5.4), starting at } z = 0, \text{ remain bounded}\}.$$

The inverse Newton transform of $u(z) = z^2 + c$ is computed by (3.1)

```
> InverseNewton(z^2+c, z)
```

giving

$$(5.6a) \quad \exp \left\{ -\frac{2}{\sqrt{4c-1}} \arctan \left(\frac{2z-1}{\sqrt{4c-1}} \right) \right\}$$

the complex version of (3.11), simplifying to

$$(5.6b) \quad \frac{z}{z-1}$$

for $c = 0$, compare with the first item in the table in Ex. 3.2.

The level sets of the absolute value of (5.6a) are shown in Fig. 7(a) for $c = 0$, and in Fig. 7(b) for $c = i$.

The number $0 \in \mathcal{M}$ since, for $c = 0$, $u(0) = 0$. This is illustrated in Fig. 7(a), showing that $z = 0$ is an attracting fixed point of the iteration $z := z^2$, or of the Newton iterates for (5.6b), which bounce off the level sets of $|z/(z-1)|$ towards $z = 0$. The point $z = 1$ is a repelling fixed point for the opposite reason.

The number i also belongs to \mathcal{M} , since the iterates of $u(z) = z^2 + i$ starting with 0 are

$$z_0 = 0, z_1 = i, z_2 = -1 + i, z_3 = -i$$

and the last 2 iterates are repeated.

These iterates correspond to the points $A(0, 0)$, $B(0, 1)$, $C(-1, 1)$, $D(0, -1)$ in Figure 7(b), each taken by the iteration (5.1) in a direction perpendicular to the level set of the absolute value of $\mathbf{N}^{-1}(z^2 + i)$ at that point. In particular the points C and D are repeated, requiring the tangents of the level sets at these points to be parallel (see dotted lines in Figure 7(b)), and an auspicious step size.

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