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# ON MATRICES OF INDEX ZERO OR ONE*

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**Notations and preliminaries.** We shall let $C^n$ denote $n$-dimensional complex vector space; $C^{m \times n}$, the $m \times n$ complex matrices; $C_r^{m \times n}$, the same with rank $r$.

For any $A \in C^{m \times n}$, let

$A^*$ denote the conjugate transpose of $A$,

$R(A)$, the range of $A$,

$N(A)$, the null space of $A$,

$I_n$, the $n \times n$ identity matrix.

For any subspace $L \subset C^n$ let $P_L$ be the perpendicular projection on $L$. For any $A \in C^{n \times n}$ define ind $A$, the index of $A$, as the smallest nonnegative integer $k$ such that rank $A^k =$ rank $A^{k+1}$.

For $A \in C^{m \times n}$ consider the matrix equations

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$$(AX)^* = AX, \tag{3}$$

$$(XA)^* = XA; \tag{4}$$

and if $m = n$, also

$$A^kXA = A^k \text{ for some integer } k > 1, \tag{$1^k$}$$

$$AX = XA. \tag{5}$$

Any $X \in C^{n \times m}$ which solves equations $(i), (j), \cdots, (l)$ from among $(1), (1^k), (2), \cdots, (5)$ is called an $\{i, j, \cdots, l\}$-*inverse* of $A$.

The set of $\{i, j, \cdots, l\}$-inverses of $A$ is denoted by $A\{i, j, \cdots, l\}$.

For any $A \in C^{m \times n}$, $A\{1, 2, 3, 4\}$ is nonempty and unique: $A\{1, 2, 3, 4\} \equiv A^\dagger$, the *Moore-Penrose generalized inverse* of $A$ (see [10], [1]).

For any $A \in C^{n \times n}$, $A\{1^k, 2, 5\}$ is nonempty if and only if $k \geqq$ ind $A$, in which case it is unique: $A\{1^k, 2, 5\} \equiv A^D$, the *Drazin pseudoinverse* of $A$ (see [4], [3], [6], [7]).

For any $A \in C^{n \times n}$, $A\{1, 2, 5\}$ is nonempty if and only if ind $A = 0$ or 1, in which case it is unique: $A\{1, 2, 5\} \equiv A^\#$, the *group inverse* of $A$ (see [5], [11], [13]). $A^\#$, whenever it exists, thus coincides with $A^D$ which exists for all $A \in C^{n \times n}$.

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$A \in C_r^{n \times n}$ is an *EPr matrix* if and only if $N(A) = N(A^*)$ or, equivalently, if and only if $A\{1, 2, 3, 4, 5\}$ is nonempty, i.e., $A^\dagger = A^\#$. EPr matrices were introduced in [12] and further studied in [8], [9].

This note deals with the class of matrices $A \in C^{n \times n}$ with $\operatorname{ind} A = 0$ or 1, i.e., $\operatorname{rank} A = \operatorname{rank} A^2$. A theorem characterizing matrices in this class is followed by a characterization of their ranges (Corollary 1) and of the subclass of EPr matrices (Corollary 2). These characterizations use limits in much the same way that nonzero scalars $\alpha$ may be characterized by the existence of $\lim_{\lambda \to 0} (\lambda + \alpha)^{-1}$.

**Results.** A characterization of the class of $A \in C^{n \times n}$ with $\operatorname{ind} A = 0$ or 1 is given in the following theorem.

THEOREM. *Let $A \in C^{n \times n}$. Then*

$$\operatorname{rank} A = \operatorname{rank} A^2 \tag{6}$$

*if and only if the limit*

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} A \tag{7}$$

*exists, in which case*

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} A = AA^\#. \tag{8}$$

*Remark.* Here $\lambda \to 0$ means $\lambda \to 0$ through any neighborhood of 0 in $C$ which excludes the nonzero eigenvalues of $-A$.

*Proof.* Let $\operatorname{rank} A = r$ and let $A$ be written as

$$A = BG, \quad \text{where } B \in C_r^{n \times r}, \quad G \in C_r^{r \times n}.$$

Cline [3] has shown that (6) is equivalent to the nonsingularity of $GB$, in which case

$$A^\# = A(GB)^{-2}G$$

and so

$$AA^\# = B(GB)^{-1}G. \tag{9}$$

It is easily verified that

$$(\lambda I_n + A)^{-1} A = B(\lambda I_r + GB)^{-1} G \tag{10}$$

whenever the inverses in question exist. The existence of the limit (7) is thus equivalent to the existence of

$$\lim_{\lambda \to 0} (\lambda I_r + GB)^{-1},$$

which, in turn, is equivalent to the nonsingularity of $GB$ and thus to (6). Equation (8) follows then from (10) and (9).

COROLLARY 1. *Let $b \in C^n$ and let $A \in C^{n \times n}$ satisfy* $\operatorname{rank} A = \operatorname{rank} A^2$. *Then $b \in R(A)$ if and only if the limit*

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} b$$

*exists, in which case*

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} b = A^{\#} b.$$

*Proof.* Writing $b \in C^n$ as

$$b = AA^{\#} b + (I_n - AA^{\#}) b,$$

we verify by using the identity

$$(\lambda I_n + A)^{-1}(I_n - AA^{\#}) = \lambda^{-1}(I_n - AA^{\#})$$

that

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} b = A^{\#} b + \lim_{\lambda \to 0} \lambda^{-1}(I - AA^{\#}) b,$$

which exists if and only if $(I - AA^{\#})b = 0$; this is equivalent to $b \in R(A)$.

If $A$ is an EPr matrix, then clearly ind $A = 0$ or 1. A characterization of EPr matrices is given in the following corollary.

COROLLARY 2. *Let $A \in C_r^{n \times n}$. Then $A$ is EPr if and only if*

$$\lim_{\lambda \to 0} (\lambda I_n + A)^{-1} P_{R(A)} = A^{\dagger}. \tag{11}$$

The proof follows from the theorem since $A^{\dagger} = A^{\#}$ if and only if $A$ is EPr. As an application of these results consider the next corollary.

COROLLARY 3 (see [2]). *Let $A \in C^{m \times n}$. Then*

$$\lim_{\lambda \to 0} (\lambda I_n + A^*A)^{-1} A^* = A^{\dagger}. \tag{12}$$

*Proof.*

$$\begin{aligned} \lim_{\lambda \to 0} (\lambda I_n + A^*A)^{-1} A^* &= \lim_{\lambda \to 0} (\lambda I_n + A^*A)^{-1} P_{R(A^*A)} A^* \\ &\qquad\qquad (\text{since } R(A^*) = R(A^*A)) \\ &= (A^*A)^{\dagger} A^* \qquad (\text{by (11) since } A^*A \text{ is EPr}) \\ &= A^{\dagger} \qquad\qquad (\text{e.g., [10]}). \end{aligned}$$

*Remark.* The limit in (12) is not restricted to real nonnegative $\lambda$ as in previous versions of this result, e.g., [1], [2].

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