A Helly-Type Theorem and Semiinfinite Programming

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The equivalence between semiinfinite convex programming (in R^n) and certain finite subprograms (with at most n constraints) is established using a Helly-type theorem due to V. Klee.

1. INTRODUCTION

A *semiinfinite program* is a mathematical program of the form

$$\text{(P)} \qquad \begin{aligned} &\inf \quad f(\mathbf{x}) \\ &\text{s.t.} \quad g(\mathbf{x}, t) \leqq 0, \qquad t \in T, \end{aligned}$$

where $\mathbf{x} \in R^n$ and T is an infinite set, which henceforth is assumed compact.

Semiinfinite programs were studied by John [15], Charnes *et al.* [3–5], Duffin and Karlovitz [8], Krabs [18], Gehner [11, 12], and others.

Under suitable assumptions, we can associate with (P) a finite subprogram

$$(\text{P} \cdot \tau) \qquad \begin{aligned} &\inf \quad f(\mathbf{x}) \\ &\text{s.t.} \quad g(\mathbf{x}, t) \leqq 0, \qquad t \in \tau, \end{aligned}$$

where τ is a finite subset of T, containing at most n elements.

If (P) is convex, this association becomes an equivalence in the sense that (P) and certain of its finite subprograms $(\text{P} \cdot \tau)$ have the same optimal solutions. This equivalence is already evident in the classical characterizations of best Tchebychev approximations [6].

If (P) is a nonconvex program, the associated finite subprograms $(\text{P} \cdot \tau)$ yield necessary conditions satisfied by the optimal solutions [15, Theorem 1].

The correspondence $(\text{P}) \leftrightarrow (\text{P} \cdot \tau)$ has been established, following John [15], by several authors, including Pshenichnyi [19, Chapter V, Corollary 2

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to Theorem 5.1], Krabs [18, Part III, Section 3.4], Laurent [14, Part II, Section 2.7], and Gehner [11, 12].

In the above derivations, the association between (P) and $(\mathrm{P}\cdot\tau)$ is obtained indirectly from optimality considerations, by using Carathéodory's theorem [2, 7].

Thus, for example, Pshenichnyi [19] replaced (P) by

$$(\tilde{\mathrm{P}}) \qquad \begin{aligned} &\inf \quad f(\mathbf{x}) \\ &\text{s.t.} \quad g(\mathbf{x}) \leqq 0, \end{aligned}$$

where

$$g(\mathbf{x}) \triangleq \sup\{g(\mathbf{x}, t) : t \in T\}.$$

Since g is, in general, nondifferentiable [even if the $g(\cdot, t)$ are], the optimality condition for $(\tilde{\mathrm{P}})$ involves the subgradient set of g, given by Valadier [21] as

$$\begin{aligned} \partial g(\mathbf{x}) &= \operatorname{cl}\operatorname{conv} \bigcup \{\partial g(\mathbf{x}, t) : t \in T(\mathbf{x})\} \\ &= \operatorname{conv} \bigcup \{\nabla g(\mathbf{x}, t) : t \in T(\mathbf{x})\}, \end{aligned}$$

under suitable differentiability assumptions, where

$$T(\mathbf{x}) = \{t \in T : g(\mathbf{x}) = g(\mathbf{x}, t)\}.$$

Using the Carathéodory theorem, an optimal $\mathbf{x}^*$ can then be characterized (under the Kuhn–Tucker condition) by the existence of $n + 1$ points $t_i \in T(\mathbf{x}^*)$ and $n + 1$ scalars $\lambda_i \geqq 0$, $i = 1, \ldots, n + 1$, satisfying

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{n+1} \lambda_i \nabla g(\mathbf{x}^*, t_i) = 0.$$

Under a suitable constraint qualification this is precisely the necessary and sufficient condition for $\mathbf{x}^*$ to be an optimal solution of $(\mathrm{P}.\tau)$, where $\tau = \{t_i : i = 1, \ldots, n + 1\}$.

The purpose of this paper is to develop the association between (P) and $(\mathrm{P}.\tau)$ directly, using a Helly-type theorem [7, 13], relating the intersection of an infinite family of convex sets to the intersections of its finite subfamilies of cardinality $\leqq n + 1$. This derivation is more elementary than the previous ones mentioned above, and it does not require differentiability. In our approach, optimality considerations enter after (P) has been reduced to a finite $(\mathrm{P}.\tau)$, permitting the use of finite optimality theory.

In Section 2 we adapt a Helly-type theorem of Klee [16, 17] for use in semiinfinite programming (Lemmas 2.2 and 2.3). Semiinfinite systems of strict convex inequalities are the subject of Corollaries 2.4 and 2.5.

In Section 3 we establish the equivalence between semiinfinite convex programs and certain finite subprograms (Theorem 3.1).

This Helly-type approach can also be used in other semiinfinite problems, e.g., semiinfinite linear inequalities [14], where Fenchel's Helly-type theorem ([10, Result 45, p. 101] or [20, Theorem 21.3]) should be applied, and Tchebychev approximations (e.g. [11, 12]). Linear equality constraints pose no additional difficulty in our Helly-type approach, since they just reduce the dimension n of the relevant space.

2. SOME CONSEQUENCES OF KLEE'S THEOREM

A family Γ of sets in R^n is called 0-*closed* if and only if every set in Γ is open and int $K \in \Gamma$ whenever K is the limit of a convergent sequence of sets of Γ [17].

Here by $\lim_{n\to\infty} K_n = K$ it is meant that

$$\bigcup_{1 \leqq m < \infty} \bigcap_{m \leqq n < \infty} K_n = \bigcap_{1 \leqq m < \infty} \bigcup_{m \leqq n < \infty} K_n = K.$$

The following Helly-type theorem for 0-closed families of convex sets will be used in the sequel.

Theorem 2.1. (Klee [17, (5.3)]). *Let Γ be a 0-closed family of convex sets. Then the intersection of all members of Γ is empty if and only if there are $n + 1$ members of Γ whose intersection is empty.*

The following lemma permits taking Γ as a finite union of 0-closed families.

Lemma 2.2. *If Γ^1, Γ^2 are 0-closed families, so is their union.*

Proof. Let $\Gamma = \Gamma^1 \cup \Gamma^2$ and let $\{K_n\} \subset \Gamma$ be a convergent sequence with $K = \lim_n K_n$. Consider the subsequences $\{K_l^1\}$ and $\{K_m^2\}$ of $\{K_n\}$, consisting of its elements in Γ^1 and Γ^2, respectively. At most one of these subsequences is finite.

Assume that $\{K_l^1\}$ is infinite, then the inclusions

$$\limsup_l K_l^1 \subset \limsup_n K_n, \qquad \liminf_l K_l^1 \supset \liminf_n K_n$$

follow easily. But the sequence $\{K_n\}$ is converged to K; hence, the above inclusions will give

$$\limsup_l K_l^1 \subset K \subset \liminf_l K_l^1,$$

which by definition implies that $\lim K_l^1 = K$. Since Γ^1 is 0-closed, it follows that int $K \in \Gamma^1 \subset \Gamma$. ■

The following lemma establishes the 0-closedness of a certain family of open level sets.

Lemma 2.3. *Let T be a compact set, $D \subset R^n$ be convex with nonvoid interior, and let $g: D \times T \to R$ be a function such that:*

(A1) *For all $\mathbf{x} \in D$, $g(\mathbf{x}, t)$ is upper semicontinuous on T.*
(A2) *For all $t \in T$, $g(\mathbf{x}, t)$ is convex on D and has the level set*

$$K_t = \{\mathbf{x} \in D \mid g(\mathbf{x}, t) < 0\}$$

nonvoid and open in D (e.g., for all $t \in T$, $g(\mathbf{x}, t)$ is upper semicontinuous on D).

Then the family of nonvoid open convex sets

$$\Gamma = \{K_t \mid t \in T\}$$

is 0-closed.

Proof. Let $\{K_{t_n}\}$ be any convergent sequence in Γ and let $\lim_n K_{t_n} = K$. We shall show that int $K \in \Gamma$, i.e., int $K = K_{t^*}$, for a certain $t^* \in T$. The compactness of T implies the existence of a subsequence $\{t_m\}$ in $\{t_n\}$ such that $\lim_m t_m = t^* \in T$. Then $\lim_m K_{t_m} = K$, due to the obvious inclusions $\limsup_m K_{t_m} \subset \limsup_n K_{t_n}$ and $\liminf_m K_{t_m} \supset \liminf_n K_{t_n}$. But

$$K_{t^*} \subset \text{int } K. \tag{1}$$

Indeed, assume $\mathbf{x} \in K_{t^*} \backslash K$. Then $g(\mathbf{x}, t^*) < 0$, while $\mathbf{x} \notin K_{t_l}$ for an infinite subsequence $\{t_l\}$ in $\{t_m\}$. Hence $g(\mathbf{x}, t_l) \geqq 0$, for every l. But $\lim_l t_l = t^*$; thus the condition (A1) will imply $g(\mathbf{x}, t^*) \geqq 0$, contradicting $g(\mathbf{x}, t^*) < 0$. Therefore $K_{t^*} \subset K$. The relation (1) will follow, since K_{t^*} is open, due to (A2). Now, we can prove the relation

$$\text{int } K = K_{t^*}. \tag{2}$$

Assume it is false. Then (1) implies

$$K_{t^*} \subset \text{int } K, \qquad K_{t^*} \neq \text{int } K,$$

which will imply the stronger relation

$$\text{int}(\text{int } K \backslash K_{t^*}) \neq \varnothing. \tag{3}$$

Indeed, take $\mathbf{z} \in \text{int } K \backslash K_{t^*}$. Since $\mathbf{z} \notin K_{t^*}$ and K_{t^*} is convex and open, there exists a hyperplane

$$H = \{\mathbf{x} \in R^n \mid h(\mathbf{x}) = a\}$$

such that $h(\mathbf{z}) = a$, while $h(\mathbf{x}) < a$ if $\mathbf{x} \in K_{t^*}$. Then int $K \cap \{\mathbf{x} \in R^n \mid h(x) > a\} \neq \varnothing$, since $\mathbf{z} \in$ int K, which implies (3). Denote

$$H_t = \{\mathbf{x} \in D \mid g(\mathbf{x}, t) = 0\}.$$

We shall prove that

$$\text{int } H_t = \varnothing, \qquad \text{for all} \quad t \in T. \tag{4}$$

Indeed, assume $\mathbf{z} \in \operatorname{int} H_t$ for a certain $t \in T$ and take $\mathbf{w} \in K_t$. Define $\gamma: R^1 \to R^1$ by

$$\gamma(\lambda) = g(\lambda \mathbf{w} + (1 - \lambda)\mathbf{z}, t), \qquad \text{for} \quad \lambda \in R^1;$$

then $\gamma(1) = g(\mathbf{w}, t) < 0$, while $\gamma = 0$ in a neighborhood of $\lambda = 0$, contradicting (A2). Now, the relations (3) and (4) will result in

$$K \setminus (K_{t^*} \cup H_{t^*}) \neq \varnothing.$$

Taking $\mathbf{x} \in K \setminus (K_{t^*} \cup H_{t^*})$ it follows that $g(\mathbf{x}, t^*) > 0$. However, $\mathbf{x} \in K = \lim_m K_{t_m}$ implies $g(\mathbf{x}, t_m) < 0$ for each m except perhaps a finite number. Since $\lim_m t_m = t^*$, the above two inequalities contradict each other according to (A1). ■

By using Lemmas 2.2 and 2.3, Klee's theorem, Theorem 2.1, is applicable to semiinfinite systems of convex inequalities.

Corollary 2.4. *Let D be a subset on R^n with nonempty interior, and $\mathscr{I}$ a finite index set. For each $i \in \mathscr{I}$ let T^i be a compact set and let $g^i: R^n \times T^i \to R$ be a function satisfying* (A1) *and* (A2) *of Lemma* 2.3. *Then the system*

$$\text{(S)} \qquad g^i(\mathbf{x}, t) < 0, \qquad t \in T^i, \quad i \in \mathscr{I},$$

has no solution $\mathbf{x} \in D$ if and only if there is a finite subset

$$\tau \subset \bigcup_{i \in \mathscr{I}} T^i$$

containing at most $(n + 1)$ *elements, such that the system*

$$(\text{S}.\tau) \qquad g^i(\mathbf{x}, t) < 0, \qquad t \in \tau \cap T^i, \quad i \in \mathscr{I}$$

has no solution in D.

Using a well-known theorem of alternatives for finite systems of convex inequalities [9], we get from Corollary 2.4 the following theorem of alternatives for infinite systems of convex inequalities.

Corollary 2.5. *The system* (S) *of Corollary* 2.4 *is inconsistent if and only if there is a finite subset*

$$\tau \subset \bigcup_{i \in \mathscr{I}} T^i$$

containing at most $(n + 1)$ *elements, and a corresponding set of nonnegative "multipliers"* $\{\lambda_t \geqq 0 : t \in \tau\}$ *not all zero, such that*

$$\sum_{i \in \mathscr{I}} \sum_{t \in \tau \cap T^i} \lambda_t g^i(\mathbf{x}, t) \geqq 0, \qquad \forall \mathbf{x} \in D.$$

A solvability theorem for semiinfinite systems of weak convex inequalities has been given by Bohnenblust *et al.* ([1, Lemma 1.5], see also [20, Theorem 21.3]).

3. SEMIINFINITE CONVEX PROGRAMS AND THEIR EQUIVALENT FINITE PROGRAMS

Consider the semiinfinite convex program

$$\text{(C)} \qquad \begin{aligned} &\inf \quad f(\mathbf{x}) \\ &\text{s.t.} \quad g^k(\mathbf{x}, t) \leqq 0, \quad t \in T^k, \quad k = 1, \ldots, m, \qquad \mathbf{x} \in D, \end{aligned}$$

where for $k = 1, \ldots, m$, T^k is a compact set, D is a convex subset of R^n with nonempty interior, and the following assumptions hold.

(A1) For each k, and for all $\mathbf{x} \in D$, $g^k(\mathbf{x}, \cdot)$ is an upper semicontinuous function on T^k.

(A2) For each k, and for all $t \in T^k$, $g^k(\cdot, t)$ is a lower semicontinuous convex function on D, and the set $\{\mathbf{x} : g^k(\mathbf{x}, t) < 0\}$ is open.

(A3) The objective function f is lower semicontinuous and convex on D.

(A4) (Slater Condition) The set

$$F^0 \triangleq \{\mathbf{x} \in D : g^k(\mathbf{x}, t) < 0, t \in T^k, k = 1, \ldots, m\},$$

is nonempty.

Theorem 3.1 *Let the convex program* (C) *satisfy* (A1)–(A4), *let* $\mathbf{x}$ *be a feasible solution of* (C) *and*

$$T^k(\mathbf{x}^*) \triangleq \{t \in T^k : g^k(\mathbf{x}, t) = 0\}, \qquad k = 1, \ldots, m,$$

$$\mathscr{P}^* \triangleq \{k : T^k(\mathbf{x}^*) \neq \varnothing\}.$$

Then $\mathbf{x}^*$ *is an optimal solution of* (C) *if and only if there is a finite set*

$$\tau^* \subset \bigcup_{k \in \mathscr{P}^*} T^k$$

containing at most n elements such that $\mathbf{x}^*$ *is an optimal solution of the finite convex problem*

$$\text{(C.}\tau^*\text{)} \qquad \begin{aligned} &\inf \quad f(\mathbf{x}) \\ &\text{s.t.} \quad g^k(\mathbf{x}, t) \leqq 0, \quad t \in \tau^* \cap T^k(x^*), \quad k \in \mathscr{P}^*, \qquad \mathbf{x} \in D. \end{aligned}$$

Proof. (i) We first show that $\mathbf{x}^*$ is optimal if and only if the system

$$\text{(B)} \qquad \begin{aligned} &f(\mathbf{x}) \quad < f(\mathbf{x}^*) \\ &\text{s.t.} \quad g^k(\mathbf{x}, t) < 0, \quad t \in T^k(\mathbf{x}^*), \quad k \in \mathscr{P}^*, \qquad \mathbf{x} \in D, \end{aligned}$$

has no solution. (See also [18, II.6.1].)

If. Suppose $\mathbf{x}^*$ is not optimal, i.e., there is a feasible $\bar{\mathbf{x}}$ with

$$f(\bar{\mathbf{x}}) < f(\mathbf{x}^*).$$

Let $\hat{\mathbf{x}} \in F^0$. Then the convex combination

$$\mathbf{x}(\lambda) = (1 - \lambda)\bar{\mathbf{x}} + \lambda\hat{\mathbf{x}}, \qquad 0 < \lambda < 1,$$

satisfies (B) for λ sufficiently small.

Only if. Suppose $\bar{\mathbf{x}}$ is a solution of (B). Then $\bar{\mathbf{x}}$ is a solution of

$$(\tilde{B}) \qquad \begin{aligned} & f(\mathbf{x}) < f(\mathbf{x}^*) \\ \text{s.t.} \quad & g^k(\mathbf{x}) < 0, \quad k \in \mathscr{P}^*, \qquad \mathbf{x} \in D, \end{aligned}$$

where

$$g^k(\mathbf{x}) \triangleq \sup\{g^k(\mathbf{x}, \ t) : t \in T^k\}.$$

Note that the supremum is attained (T^k is compact, and $g^k(\mathbf{x}, \cdot)$ is upper semicontinuous), so that

$$g^k(\mathbf{x}) = g^k(\mathbf{x}, t)$$

for some $t \in T^k$. Thus, in particular,

$$g^k(\mathbf{x}^*) < 0 \qquad \text{for} \quad k \notin \mathscr{P}^*.$$

Consider the convex combination

$$\mathbf{x}(\lambda) \triangleq (1 - \lambda)\mathbf{x}^* + \lambda\bar{\mathbf{x}}, \qquad 0 < \lambda < 1.$$

Then $\mathbf{x}(\lambda) \in D$, by the convexity of D. Further, by the convexity of g^k,

$$g^k(\mathbf{x}(\lambda)) \leqq (1 - \lambda)g^k(\mathbf{x}^*) + \lambda g^k(\bar{\mathbf{x}}) \begin{cases} < 0, & k \in \mathscr{P}^* \\ < 0, & k \notin \mathscr{P}^* \end{cases}$$

if λ is sufficiently small, and by the convexity of f,

$$f(\mathbf{x}(\lambda)) \leqq (1 - \lambda)f(\mathbf{x}^*) + \lambda f(\bar{\mathbf{x}}) < f(\mathbf{x}^*).$$

Thus, for λ sufficiently small, $\mathbf{x}(\lambda)$ is a feasible solution of (C), with

$$f(\mathbf{x}(\lambda)) < f(\mathbf{x}^*),$$

contradicting the optimality of x^*.

(ii) Denote $g^0(x, t) \triangleq f(x) - f(x^*)$, $T^0 \triangleq \{0\}$.

By a straightforward application of Corollary 2.4, we conclude that the system (B) is inconsistent if and only if there is a finite subset

$$\tau^* \subset \bigcup_{k \in \mathscr{P}^* \cup \{0\}} T^k$$

containing at most $n + 1$ elements such that the following finite system is inconsistent:

$$(\text{B}.\tau^*) \qquad g^k(\mathbf{x}, t) < 0, \qquad t \in \tau^* \cap T^k(\mathbf{x}^*), \quad k \in \mathscr{P}^* \cup \{0\}, \quad \mathbf{x} \in D.$$

We claim now that $0 \in \tau^*$. Indeed, if $0 \notin \tau^*$, then the nonexistence of a solution to the system

$$g^k(\mathbf{x}, t) < 0, \qquad t \in \tau^* \cap T^k(\mathbf{x}^*), \quad k \in \mathscr{P}^*,$$

will contradict (by the "if" part of Corollary 2.4) assumption (A4). We conclude that there is a finite subset $\tau^* \subset \bigcup_{k \in \mathscr{P}^*} T^k$ containing at most n elements such that the system

$$(\mathrm{B}.\tau^*) \qquad \begin{aligned} & f(\mathbf{x}) < f(\mathbf{x}^*) \\ \text{s.t.} \quad & g^k(\mathbf{x}, t) < 0, \quad t \in \tau^* \cap T^k(\mathbf{x}^*), \quad k \in \mathscr{P}^*, \qquad \mathbf{x} \in D, \end{aligned}$$

has no solution.

(iii) Similarly to (i) we can now establish that the inconsistency of (B.τ^*) is equivalent to $\mathbf{x}^*$ being an optimal solution of $(P.\tau^*)$. ■

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