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Input Optimization for Infinite-Horizon Discounted Programs[1,2]

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Abstract. Let $V(x)$ denote the optimal (maximal) value of a discrete time dynamic program, given the initial state x. This paper is concerned with the inverse correspondence V^{-1} and its infimum

$$I(v) := \inf_{x} \{x \colon V(x) \geq v\}$$

for discounted, infinite-horizon programs with 1-dimensional state space and monotone $V(\cdot)$. The function $I(v)$, interpreted as the optimal (minimal) input required to achieve v, is computed using dynamic programming recursion (Theorem 3.1) or value iteration (Theorem 3.2). An application to mathematical economics (optimal consumption plan) is given.

Key Words. Dynamic programming, inverse dynamic programming, infinite-horizon problems, optimality principle, value iteration, optimal plans.

1. Introduction

A decision process can be described in terms of its input (e.g., resources), output (e.g., performance criterion), and the decisions connecting them. In this context there are two natural problems:

(P) Primal Problem. Given an input x, find the optimal value $V(x)$ of the output (e.g., maximal profit) and decisions leading to it (called optimal decisions); see, e.g., Ref. 1.

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(D) Dual Problem. Given an output level v, find an optimal input (or valuation of an optimal input) $I(v)$, commensurable with v, and the corresponding optimal decisions.

Duality theory, the study of relations between (P) and (D) and their optimal decisions and values, is an important part of linear and nonlinear programming, underlying algorithms and sensitivity analysis. The role of duality in dynamic programming is less pronounced, mainly due to the recursive computations and the need to compute the optimal value functions over their domains. Duality in dynamic programming, indicated in Ref. 2 (Chapter 5) and used algorithmically in Ref. 3 and elsewhere, has been studied systematically by Iwamoto (in particular, Refs. 4 and 5), emphasizing the inverse relations between the functions $V(\cdot)$ and $I(\cdot)$, and their recursive computations.

This paper is concerned with discrete-time infinite-horizon discounted dynamic programs with one-dimensional state space, and the properties and computation of the optimal input $I(\cdot)$, extending the finite-horizon results of Ref. 6.

The model and hypotheses used are described in Section 2. Section 3 gives the main results on input optimization, the inverse relations between $V(\cdot)$ and $I(\cdot)$ and their continuity properties (Proposition 3.1), an optimality principle for the optimal input (Theorem 3.1), and a value-iteration computation of $I(\cdot)$ (Theorem 3.2). Section 4 concludes with an example in mathematical economics, where we actually compute the optimal input.

2. Model

Consider a process that is observed at time t to be in the state $x_t \in \mathbf{X} \subseteq \mathcal{R}$, $t = 0, 1, 2, \ldots$, the initial point $x_0 = x$ specified. After observing the current state, a control $u_t \in U_t(x_t)$ is chosen resulting in an immediate reward $r_t(x_t, u_t)$ and a new state

$$x_{t+1} = f_t(x_t, u_t). \tag{1}$$

The objective is to maximize the overall value

$$\sum_{t=0}^{T} r_t(x_t, u_t)$$

of the flow of rewards from time 0 up to the planning horizon T.

When T is finite, the dynamic programming approach uses the optimal value functions V_τ^T, $\tau = 0, \ldots, T$, defined on $\mathbf{X}$ by

$$V_\tau^T(x_\tau) := \max_{u_t \in U_t(x_t)} \sum_{t=\tau}^{T} r_t(x_t, u_t), \tag{2}$$

with x_τ fixed, and the dynamics governed by (1) for all $t \geq \tau$.

The *boundary condition*

$$V_T^T(x_T) = \max_{u_T \in U_T(x_T)} r_T(x_T, u_T), \tag{3}$$

together with the backward recursion

$$V_t^T(x_t) = \max_{u_t \in U_t(x_t)} \{r_t(x_t, u_t) + V_{t+1}^T(f_t(x_t, u_t))\}, \tag{4}$$

for $t = T-1, T-2, \ldots, 0$, yields finally the optimal value

$$V^T(x_0) = V_0^T(x_0), \qquad \text{for } x_0 = x, \tag{5}$$

and the optimal decisions $u_0, \ldots, u_T$ furnishing this value.

Some restrictions must be imposed on the data in order to make definitions (2), (3) and the relation (4) well defined. To this end, the following assumption will be in force throughout the paper.

Assumption 2.1. Regularity. (i) The state space $\mathbf{X}$ is a closed subset of $\mathcal{R}$.

(ii) For each $t \geq 0$, the point-to-set mapping U_t from $\mathbf{X}$ into the topological space $\mathbf{U}$ is upper semicontinuous (Ref. 7, Chapter 3), and $U_t(x)$ is nonempty compact for each $x \in \mathbf{X}$.

(iii) The transition function $f_t : \mathbf{X} \times \mathbf{U} \to \mathbf{X}$ is continuous (in the product topology) for each $t \geq 0$.

(iv) The reward function $r_t : \mathbf{X} \times \mathbf{U} \to \mathcal{R}$ is bounded and upper semicontinuous for each $t \geq 0$.

Then, it is straightforward to verify that the maxima in (2) and (3) are indeed attained and, moreover, $V_t^T(\cdot)$ is upper semicontinuous for each t, $0 \leq t \leq T$; see, e.g., Ref. 7, Proposition 3.1.21.

When T is infinite, some provision is needed to make the criterion $\sum_{t=0}^{\infty} r_t(x_t, u_t)$ proper. Toward this, we assume the following stationarity properties of the model.

Assumption 2.2. Stationarity. When $T = \infty$, we shall suppose that, for all $t \geq 0$:

(i) $f_t = f$, with f continuous;

(ii) $r_t = \delta^t r$, with $\delta \in (0, 1)$ being a discount factor and r bounded upper semicontinuous;

(iii) $U_t = U$, where $U(\cdot)$ is upper semicontinuous with nonempty compact values.

The infinite-horizon optimal value function $V(x_0)$ is defined as

$$V(x_0) := \sup_{\pi \in \Pi} \left\{ \lim_{T \to \infty} V_\pi^T(x_0) \right\},$$

where Π is the set of all possible infinite-stage policies and $V_\pi^T(x_0)$ is the T-period return of policy π with initial state x_0. For our problem, this optimal value function may be obtained as

$$V(x_0) = \lim_{T\to\infty} V^T(x_0),$$

where $V^T(x_0)$ is the optimal return of a T-period problem defined by relations (2)–(5). In fact, $V(\cdot)$ will also result from *value iteration*; that is, for any bounded, upper semicontinuous function $V^{(0)}: \mathbf{X} \to \mathscr{R}$, the sequence of functions $V^{(T)}$, $T = 1, 2, \ldots$, defined on $\mathbf{X}$ by

$$V^{(T+1)}(x_0) := \max_{u_0 \in U(x_0)} \{r(x_0, u_0) + \delta V^{(T)}(f(x_0, u_0))\}, \tag{6}$$

will converge uniformly to V.

For proof of these results in much greater generality, see Ref. 8, Theorem 1.13. We emphasize that the limit function V inherits the property of upper semicontinuity. Note that, if

$$V^{(0)}(x) := \max_{u \in U(x)} r(x, u),$$

then $V^{(T)} = V^T$.

So far we have not relied on any of the order structures of the state space $\mathbf{X}$. From here on, however, we shall suppose that the model enjoys some monotonicity properties.

Assumption 2.3. Monotonicity. For any $0 \le t \le T$, $V_t^T(\cdot)$ and $V(\cdot)$ are monotone nondecreasing functions.

We remark that this assumption is satisfied if, for every $t \ge 0$ and $u \in U$, the functions $f_t(x, u)$, $r_t(x, u)$, $f(x, u)$, $r(x, u)$ and the sets $U_t(x)$, $U(x)$ are monotone nondecreasing in x, the latter meaning that

$$U_t(x_1) \subset U_t(x_2), \quad U(x_1) \subset U(x_2), \qquad \text{whenever } x_1 \le x_2.$$

In economic terms, Assumption 2.3 means that we allow for free disposal of resources at any stage.

3. Input Optimization

We are concerned with the computation of the functions

$$I_t^T(v) := \inf\{x_t : V_t^T(x_t) \ge v\},$$

associated with the finite-horizon models, and the corresponding function

$$I(v) := \inf\{x_0 : V(x_0) \ge v\},$$

in case the horizon is infinite.

These functions report on the minimal input needed to produce at least the value v during the periods to go. Without any qualifications on V_t^T, V, we may ascertain that I_t^T, I are monotone nondecreasing, provided the convention $\inf \phi = +\infty$ is used to reflect infeasibility.

It is also clear that

$$I_t^T(V_t^T(x_t)) \leq x_t, \qquad \text{for all } t,\ x_t \in \mathbf{X}, \tag{7}$$

$$I(V(x_0)) \leq x_0. \tag{8}$$

However, there is more to it.

Proposition 3.1. Under Assumption 2.1, for all $t \leq T$,

$$V_t^T(I_t^T(v)) \geq v, \tag{9}$$

$$I_t^T(v) = \min\{x_t :\ V_t^T(x_t) \geq v\}, \tag{10}$$

and I_t^T is lower semicontinuous.

Under Assumptions 2.1 and 2.3,

$$I_t^T(v) = \min\{x_t :\ V_t^T(x_t) = v\}, \qquad \text{for all } v \in \text{range}\, V_t^T,$$

that is,

$$I_t^T = \min(V_t^T)^{-1}, \qquad \text{on range } V_t^T.$$

Similarly, under Assumptions 2.1 and 2.2,

$$V(I(v)) \geq v,$$

$$I(v) = \min\{x_0 :\ V(x_0) \geq v\},$$

and I is lower semicontinuous.

If Assumption 2.3 is also in force, then

$$I = \min V^{-1}, \qquad \text{on range } V.$$

Proof. The set $\{x_t :\ V_t^T(x_t) \geq v\}$ is closed by the upper semicontinuity of V_t^T. Thus, its greatest lower bound $I_t^T(v)$ satisfies (9), and now (10) follows immediately.

For the lower semicontinuity of I_t^T, let $v^n \to v$, and suppose that

$$\liminf_{n\to\infty} I_t^T(v^n) = \lim_{k \in K} I_t^T(v^k),$$

for some subsequence K. Then, by the upper semicontinuity of V_t^T and (9),

$$V_t^T(\liminf_{n\to\infty} I_t^T(v^n)) \geq \limsup_{k \in K} V_t^T(I_t^T(v^k)) \geq \limsup_{k \in K} v^k = v.$$

Hence,

$$I_t^T(v) \le \liminf_{n\to\infty} I_t^T(v^n),$$

and this takes care of the lower semicontinuity of I_t^T. The statements involving I are proved similarly. □

Having explored the properties of I_t^T and I, we are ready to show that these functions satisfy an optimality principle of dynamic programming.

Theorem 3.1. (i) Under Assumptions 2.1 and 2.3, if $t < T$, then

$$I_t^T(v) = \min\{x_t : f_t(x_t, u_t) \ge I_t^T(v - r_t(x_t, u_t)), \quad \text{for some } u_t \in U_t(x_t)\}.$$

(ii) Similarly, under Assumptions 2.1, 2.2, and 2.3,

$$I(v) = \min\{x_0 : f(x_0, u_0) \ge I(\delta^{-1}(v - r_t(x_t, u_t)), \quad \text{for some } u_0 \in U(x_0)\}.$$

Proof. (i) $V_t^T(x_t) \ge v$, iff

$$V_{t+1}^T(f_t(x_t, u_t)) \ge v - r_t(x_t, u_t), \qquad \text{for some } u_t \in U_t(x_t). \tag{11}$$

Now, apply I_{t+1}^T on both sides of (11) to obtain

$$f_t(x_t, u_t) \ge I_{t+1}^T(v - r_t(x_t, u_t)), \qquad \text{for some } u_t \in U_t(x_t). \tag{12}$$

Here, we relied on (7) and the monotonicity of I_{t+1}^T. Conversely, apply V_{t+1}^T on both sides of (12) and invoke (9) to regain (11). The arguments supporting (ii) are entirely similar. □

We remark that assertion (i) above has already been established in Ref. 6. It also follows from the more general results in Ref. 4, Section 3.

We conclude this section by showing that, when $T = +\infty$, the optimal input will result from a process corresponding to value iteration. To avoid nonessential technicalities, we assume henceforth that $\mathbf{X}$ is connected and not bounded above; see also the remark following the proof of Theorem 3.2.

Theorem 3.2. Input Iteration. Let Assumptions 2.1 and 2.2 be satisfied. Suppose that $U(x)$ and $f(x, u)$ are nondecreasing and $r(x, u)$ is increasing in x for each u. For an arbitrary bounded, upper semicontinuous, monotone nondecreasing $V^{(0)} : \mathbf{X} \to \mathscr{R}$, define

$$I^{(0)}(v) := \min\{x_0 : V^{(0)}(x_0) \ge v\},$$

and proceed by setting

$$I^{(T+1)}(v) := \min\{x_0 : f(x_0, u_0) \geq I^{(T)}(\delta^{-1}(v - r(x_0, u_0))),$$
$$\text{for some } u_0 \in U(x_0)\},$$

for $T = 0, 1, 2, \ldots$.

Then, for every $v \in \mathcal{R}$, each cluster point of $I^{(T)}(v)$, $T = 0, 1, \ldots$, equals $I(v)$. In particular, if the optimal value function V is *sup-compact*, in the sense that $\{x\colon V(x) \geq v\}$ is compact, then $I^{(T)}(v) \to I(v)$.

Proof. For $T = 0, 1, \ldots$, define recursively $V^{(T+1)}$ on $\mathbf{X}$ by (6). Then, $V^{(T)}$ converges uniformly to V as $T \to \infty$.

All the functions $V^{(T)}$ are monotone nondecreasing and upper semicontinuous. Suppose inductively that

$$I^{(T)}(v) = \min\{x_0 \colon V^{(T)}(x_0) \geq v\}. \tag{13}$$

Then, $I^{(T)}$ is monotone nondecreasing and satisfies

$$I^{(T)}(V^{(T)}(x_0)) \leq x_0, \tag{14}$$

$$V^{(T)}(I^{(T)}(v)) \geq v. \tag{15}$$

Here, (14) is self-evident from (13), and (15) follows from the upper semicontinuity of $V^{(T)}$. Now, by (14), (15), and the monotonicity of $V^{(T)}$, $I^{(T)}$, the inequality

$$f(x_0, u_0) \geq I^{(T)}(\delta^{-1}(v - r(x_0, u_0))), \qquad \text{for some } u_0 \in U(x_0), \tag{16}$$

can be stated equivalently as

$$V^{(T)}(f(x_0, u_0)) \geq \delta^{-1}(v - r(x_0, u_0)), \qquad \text{for some } u_0 \in U(x_0). \tag{17}$$

Clearly, (17) amounts to

$$V^{(T+1)}(x_0) \geq v. \tag{18}$$

Thus, the equivalence of (16) and (18) implies that

$$I^{(T+1)}(v) = \min\{x_0 \colon V^{(T+1)}(x_0) \geq v\}.$$

Now, fix $v \in \mathcal{R}$, and introduce the functions

$$L^T(x_0) := \begin{cases} x_0, & \text{if } V^{(T)}(x_0) \geq v, \\ +\infty, & \text{otherwise,} \end{cases}$$

for $T = 0, 1, 2, \ldots$. The function L is defined similarly in terms of V. It is easy to see that L^T, L are lower semicontinuous. We assert that (a) and (b) below hold.

(a) For any $x_0 \in \mathbf{X}$, if $x_0^T \to x_0$, then

$$\liminf_{T\to\infty} L^T(x_0^T) \geq L(x_0).$$

To see this, it suffices to consider the case when

$$\liminf_{T\to\infty} L^T(x_0^T) < +\infty.$$

Then,

$$\liminf_{T\to\infty} L^T(x_0^T) = x_0,$$

and we must show that $V(x_0) \geq v$. If not, then for some $\epsilon > 0$,

$$V(x_0) < v - 3\epsilon.$$

By the upper semicontinuity of V, we may find a neighborhood N of x_0 such that

$$V(y_0) \leq V(x_0) + \epsilon,$$

whenever $y_0 \in N$. Since $V^{(T)} \to V$ uniformly, there exists an integer τ_1 such that

$$V^{(T)} \leq V + \epsilon, \qquad \text{for all } T \geq \tau_1.$$

For $T >$ some integer τ_2, we have $x_0^T \in N$. Then, for $T \geq \tau := \max\{\tau_1, \tau_2\}$,

$$V^{(T)}(x_0^T) \leq V(x_0^T) + \epsilon \leq V(x_0) + 2\epsilon < v.$$

Thus,

$$L^T(x_0^T) = +\infty, \qquad \text{for all } T \geq \tau.$$

This, however, contradicts the finiteness of

$$\liminf_{T\to\infty} L^T(x_0^T).$$

(b) For any $x_0 \in \mathbf{X}$, there exists a sequence $x_0^T \to x$ such that

$$\limsup_{T\to\infty} L^T x_0^T \leq L(x_0).$$

If $L(x_0) = +\infty$, there is nothing to prove. Otherwise, pick a decreasing sequence $x_0^n \to x_0$. Note that V is monotone increasing. Thus, $V(x_0^n) > V(x_0) \geq v$, for all n. Since $V^{(T)} \to V$ pointwise, we have that $V^{(T)}(x_0^n) \geq v$, for all T sufficiently large; that is, $L^T(x_0^n) = x_0^n$, for large T. Thus,

$$\limsup_{n\to\infty} \limsup_{T\to\infty} L^T(x_0^n) = x_0 = L(x_0).$$

Now, for some sequence $n(T)$, we have

$$\limsup_{T\to\infty} L^T(x_0^{n(T)}) \leq \limsup_{n\to\infty} \limsup_{T\to\infty} L^T(x_0^n) = L(x_0);$$

see Ref. 9, Corollary 1.16. This takes care of (b).

To complete the proof, let x_0 be a cluster point of the sequence $x^T = I^{(T)}(v)$, $T = 0, 1, 2, \ldots$. That is, for some subsequence K,

$$x_0 = \lim_{k \in K} x_0^k.$$

Let y_0 be any other element of $\mathbf{X}$. For some sequence $y_0^T \to y_0$, we have

$$\limsup_{T\to\infty} L^T(y_0^T) \le L(y_0), \qquad \text{by (b)}.$$

Let $x_0^T = x_0$, whenever $T \notin K$. Then, (a) implies that

$$\begin{aligned} L(x_0) &\le \liminf_{T\to\infty} L^T(x_0^T) \le \liminf_{k\in K} L^k(x_0^k) \\ &\le \limsup_{k\in K} L^k(y_0^k) \le \limsup_{T\to\infty} L^T(y_0^T) \le L(y_0). \end{aligned}$$

This shows that x_0 minimizes L. Hence, $x_0 = I(v)$, ending the proof. □

Remark 3.1. A closer scrutiny of the preceding proof shows that it is not necessary to assume $\mathbf{X}$ connected and unbounded from above. It suffices that, for each $x \in \mathbf{X}$, there exists a sequence $x^n \to x$ such that $V^T(x^n) \ge V(x)$ when T is sufficiently large. Toward this, if x belongs to the closure of $\mathbf{X} \cap (x, +\infty)$, we can take $x^n \downarrow x$ in $\mathbf{X}$. Otherwise, make sure that $V^{(0)} > \sup r$, and let $x^n \equiv x$.

4. Example

The successive approximations of $I(v)$ by the recursion

$$I^{(T+1)}(v) = \min\{x_0 : f(x_0, u_0) \ge I^{(T)}(\delta^{-1}(v - r(x_0, u_0)))\}, \qquad \text{for some } u_0 \in \mathbf{U},$$

is in general rather cumbersome. Yet, this scheme may entail no more difficulties than (6), as shown by the following example based on Ref. 10.

Let $x_0 \ge 0$, $r(x, u) = u^\alpha$, with $0 < \alpha < 1$, $U(x) = [0, x]$, and $f(x, u) = \gamma(x - u)$, with $\gamma > 0$.

Initiate the input iteration of Theorem 3.2 by letting

$$V^{(0)}(x) = x^\alpha.$$

Then, for $v \ge 0$,

$$I^{(0)}(v) = \min\{x : x^\alpha \ge v\} = v^\beta,$$

with $\beta = \alpha^{-1}$. Next, we have

$$\begin{aligned} I^{(1)}(v) &= \min\{x : \gamma(x-u) \ge (\delta^{-1}(v - u^\alpha))^\beta, \text{ for some } u \in [0, x]\} \\ &= \min\{x : x = u + \gamma^{-1}(\delta^{-1}(v - u^\alpha))^\beta,\ u \in [0, v^\beta]\} \\ &= b_1 v^\beta, \end{aligned}$$

with

$$b_1 = \frac{1+\gamma^{-1}(\delta\gamma)^{1/(1-\alpha)}}{[1+\delta(\delta\gamma)^{\alpha/(1-\alpha)}]^\beta}, \qquad u = \left[\frac{v}{1+\delta(\delta\gamma)^{\alpha/(1-\alpha)}}\right]^\beta,$$

as follows by routine calculus. This suggests a trial solution

$$I^{(T)}(v) = b_T v^\beta,$$

which is indeed verified by simple analysis. Moreover,

$$b_{T+1} = \frac{1+b_T\gamma^{-1}(\delta_\gamma b_T^{-1})^{1/(1-\alpha)}}{[1+\delta(\delta\gamma b_T^{-1})^{\alpha/(1-\alpha)}]^\beta}.$$

It remains to identify $\lim_{T\to\infty} b_T$. Toward this, introduce

$$\beta_T = b_T^{-\alpha/(1-\alpha)} = b_T^{1/(1-\beta)},$$

$$\omega = (\delta^\alpha\gamma)^{1/(1-\alpha)}.$$

Then,

$$b_{T+1} = (1+\beta_T^{1-\beta}\omega\beta_T^\beta)/(1+\omega\beta_T)^\beta = 1/(1+\omega\beta_T)^{\beta-1}.$$

Hence,

$$\beta_{T+1} = 1+\omega\beta_T.$$

Recall that $\beta_0 = 1$. Thus,

$$\beta_T = \sum_{t=0}^{T} \omega^t,$$

$$b_T = \left[\sum_{t=0}^{T} \omega^t\right]^{\alpha-1/\alpha}.$$

It follows that $\lim_{T\to\infty} b_T$ exists if and only if $\omega < 1$; that is, we must have

$$\gamma^\alpha < \delta^{-1} := 1+i, \tag{19}$$

with $i > 0$ being the interest rate.

The economic interpretation of (19) is that the opportunity cost of capital (or the rate of impatience) exceeds the benefits of deferred consumption.

Note that, unless $\gamma \le 1$, we cannot *a priori* guarantee that $r(x, u)$ is bounded above. This and similar examples (Ref. 11) provide strong motivation for pushing beyond Assumption 2.1(iv). Then, growth conditions must be imposed on the reward functions, and more care is needed when designing the finite-horizon approximations (Ref. 12).

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